Amenability of quasi-lattice ordered groups

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Overview

1. Quasi-lattice ordered groups: definition and examples
2. Isometries
3. C^*-algebras of quasi-lattice ordered groups
4. Amenability of quasi-lattice ordered groups

Let P be a generating submonoid of a discrete group G st $P \cap P^{-1} = \{e\}$. For $x, y \in G$ define

$$x \leq y \iff x^{-1}y \in P \iff y \in xP.$$

Then $\leq$ is a partial order on G.

Definition (Nica, 1992)

(G, P) is *quasi-lattice ordered* if all $x, y \in G$ with a common upper bound in P have a least common upper bound $x \vee y$ in P.
Equivalently,

1. Any $x \in PP^{-1}$ has a least upper bound in P; and
2. any $x, y \in P$ with a common upper bound have a least common upper bound.

(Item (1) implies (2).)
Write $x \vee y < \infty$ if the least upper bound exists, $x \vee y = \infty$ else.

From above: (G, P) is *quasi-lattice ordered* if all $x, y \in G$ with a common upper bound in P have a least common upper bound $x \vee y$ in P.

Examples

1. In $(\mathbb{Z}^2, \mathbb{N}^2)$, have $(m_1, m_2) \leq (n_1, n_2)$ iff $m_i \leq n_i$. So $(m_1, m_2) \vee (n_1, n_2) = (\max(m_1, n_1, 0), \max(m_2, n_2, 0))$.
2. $(\mathbb{Q}_+^*, \mathbb{N}^\times)$ under multiplication. For $m, n \in \mathbb{N}^\times$, $m \leq n$ means $m \mid n$, and $m \vee n$ is the lowest common multiple.
3. Let G be the free group with generators a, b, and P the submonoid consisting of words in a and b. Then $x \leq y$ means that x is an initial segment of y, and the rest of y has no factors of a^{-1} or b^{-1}. Here $x \vee y = \infty$ often.

Motivating example

Let $c, d \in \mathbb{N}^+$. The *Baumslag-Solitar group* has presentation

$$G = \langle a, b : ab^c = b^d a \rangle.$$

Let P be the submonoid generated by a and b. Then (G, P) is quasi-lattice ordered (Spielberg, 2012). Crucial is that G is an *HNN extension* of $\mathbb{Z}$, and hence each $x \in G$ has a unique *normal form*. Write $\theta : G \to \mathbb{Z}$ for the homomorphism such that $\theta(a) = 1$ and $\theta(b) = 0$. If $x \in P$, then

$$x = b^{s_0} ab^{s_1} \cdots b^{s_{k-1}} ab^{s_k}.$$

where $k = \theta(x)$ and each $0 \leq s_{i-1} < d$. We call θ the *height map*.

- $a^2 b^{42} = b^0 ab^0 ab^{42}$
- $b^d a = ab^c = b^0 ab^c$
- If $0 \leq n < d$, then $b^{n+d} a = b^n ab^c$.

Spielberg's proof that the B-S group is quasi-lattice ordered uses that (G, P) is quasi-lattice ordered iff (Crisp-Laca, 2002)

> if $x \in PP^{-1}$, then there exist a pair $\mu, \nu \in P$ with $x = \mu\nu^{-1}$ such that $\gamma, \delta \in P$ and $\gamma\delta^{-1} = \mu\nu^{-1}$ imply $\mu \leq \gamma$ and $\nu \leq \delta$. (The pair μ, ν is unique.)

Tease: Thompsen's group

$$F = \left\langle \{x_i : i \in \mathbb{N}\} : x_i x_k = x_k x_{i+1} \text{ for } i > k \right\rangle.$$

Let P be the submonoid generated by $\{x_i : i \in \mathbb{N}\}$. Then (F, P) is quasi-lattice ordered (Nucinkis, 2017). In fact, (F, P) is lattice ordered.

Let H be a Hilbert space. A linear bounded operator S on H is an *isometry* if $\|Sh\| = \|h\|$ for all $h \in H$. Equivalently, $S^*S = 1$.

Example

$H = \ell^2(\mathbb{N})$, the square-summable sequences. The unilateral shift S

$$S(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$$

is an isometry with *adjoint* $S^*(x_0, x_1, \dots) = (x_1, x_2, \dots)$. The *range projection* SS^* is the orthogonal projection onto the subspace of sequences with a zero in the first entry.

An *isometric representation* of a monoid P is a homomorphism $S : P \to B(H)$ such that each S_p is an isometry.

Example

Consider $\ell^2(P)$ with o.n. basis $\{e_x : x \in P\}$. For each $x \in P$, there is an isometry T_x on $\ell^2(P)$ such that $T_x e_y = e_{xy}$ for $y \in P$. Then T is an isometric representation of P.

Suppose that (G, P) is a quasi-lattice ordered group. Let $T : P \to \ell^2(P)$ be $T_x e_y = e_{xy}$ for $y \in P$ as above. T is called the *Toeplitz representation*. Nica observed that

$$T_x T_x^* T_y T_y^* = \begin{cases} T_{(x\vee y)} T_{(x\vee y)}^* & \text{if } x \vee y < \infty \\ 0 & \text{if } x \vee y = \infty. \end{cases} \tag{1}$$

Equivalently, $T_x^* T_y = \begin{cases} T_{x^{-1}(x\vee y)} T_{y^{-1}(x\vee y)}^* & \text{if } x \vee y < \infty \\ 0 & \text{if } x \vee y = \infty. \end{cases}$

An isometric rep of P satisfying (1) is *Nica-covariant*.

(G, P) has two C^*-algebras:

1. The *Toeplitz algebra* is $\mathcal{T}(P) := \langle T_x : x \in P \rangle \subset B(\ell^2(P))$.
2. The C^*-algebra $C^*(G, P) = \overline{\text{span}}\{w_x w_y^* : x, y \in P\}$ universal for Nica-covariant reps. (Here w is a universal Nica-covariant rep.)

The Toeplitz representation $T : P \to \mathcal{T}(P)$ induces a surjection $\pi_T : C^*(G, P) \to \mathcal{T}(P)$.

Example: Baumslag-Solitar $G = \langle a, b : ab^c = b^d a \rangle$.

Thm. (Clark-aH-Raeburn, 2016)
$C^*(G, P)$ is universal for isometries U and V satisfying

1. $VU^c = U^d V$;
2. $U^* V = U^{d-1} V U^{*c}$;
3. $V^* U^j V = 0$ for $1 \leq j < d$.

It's easy to see that the Toeplitz representation satisfies (1)-(3) with $U := T_b$ and $V := T_a$ in $B(l^2(P))$.

1. $VU^c = T_a T_b^c = T_{ab^c} = T_{b^d a} = T_{b^d} T_a = U^d V$.
2. Since $a \vee b = ab^c = b^d a$, Nica covariance gives

$$U^* V = T_b^* T_a = T_{b^{-1}(a \vee b)} T^*_{a^{-1}(a \vee b)} = T_{b^{d-1} a} T^*_{b^c} = U^{d-1} V U^{*c}.$$

3. Let $1 \leq j < d$. Then $a \vee b^j a = \infty$, Nica covariance says $V^* U^j V = 0$.

Say (G, P) is *amenable* if the Toeplitz rep induces an isomorphism of the universal C^*-algebra $C^*(G, P)$ onto the Toeplitz C^*-algebra.

- There exists a linear idempotent $E : C^*(G, P) \to \overline{\text{span}}\{w_p w_p^* : p \in P\}$. Then (G, P) is amenable if and only if $E(a^* a) = 0$ implies $a = 0$.
- There are Følner-type conditions for amenability.

Tease:

Thompson's group F is amenable if and only if (F, P) is amenable as a quasi-lattice ordered group.

Examples of amenable (G, P)

- (Nica) If G is an amenable group, then (G, P) is an amenable quasi-lattice ordered group.
- (Nica) Let $\mathbb{F}_2$ be the free group on $\{a, b\}$ and let $\mathbb{F}_2^+ = \langle a, b \rangle$. Then $(\mathbb{F}_2, \mathbb{F}_2^+)$ is amenable.
- (Clark-aH-Raeburn, 2016) The Baumslag-Solitar (G, P) is amenable. To prove this we had to generalise the proof of:

Theorem (Laca-Raeburn, 1996)

Let (G, P) and (K, Q) be quasi-lattice ordered groups and let $\mu : G \to K$ be "controlled". If K is amenable, then (G, P) is.

Controlled means: μ is an order-preserving group homomorphism such that for $x, y \in P$ with $x \vee y < \infty$:

1. $\mu(x) \vee \mu(y) = \mu(x \vee y)$;
2. $\mu(x) = \mu(y) \Rightarrow x = y$.

Note: $\mu^{-1}(e) \cap P = \{e\}$.

The Baumslag-Solitar height map θ is not controlled since $\theta^{-1}(e) \cap P = \{b^n : n \in \mathbb{N}\}$.

We used ideas from Clark-aH-Raeburn about the B-S group to find a new notion of "controlled map" and then used the strategy from Laca-Raeburn to show amenability.

Theorem (aH-Raeburn-Tolich, preprint 2017)

Suppose that $\mu : (G, P) \to (K, Q)$ is controlled (in a new technical sense).

1. $(\mu^{-1}(e), \mu^{-1}(e) \cap P)$ is a quasi-lattice ordered group.
2. If K is an amenable group and $(\mu^{-1}(e), \mu^{-1}(e) \cap P)$ is an amenable quasi-lattice group, then (G, P) is amenable.

The idea for this comes from:

Theorem (Spielberg, 2014)

Let G be a Hausdorff étale groupoid, let Q be a countable abelian group, and let $c : G \to Q$ be a continuous homomorphism. Then

1. $c^{-1}(0)$ is a Hausdorff étale groupoid.
2. If $c^{-1}(0)$ is an amenable groupoid, then G is.

The new notion of "controlled" map

Let $\mu : G \to K$ be an order-preserving group homomorphism. Then μ is *controlled* if it has the following properties:

1. For $x, y \in P$ with $x \vee y < \infty$, we have $\mu(x) \vee \mu(y) = \mu(x \vee y)$.
2. Let $q \in Q$. Set $\Sigma_q = \{\sigma \in \mu^{-1}(q) \cap P : \sigma \text{ is minimal}\}$.
   a. If $x \in \mu^{-1}(q) \cap P$, then there exists $\sigma \in \Sigma_q$ such that $\sigma \leq x$.
   b. If $\sigma, \tau \in \Sigma_q$, then $\sigma \vee \tau < \infty \Rightarrow \sigma = \tau$.

Examples

- Let G be the Baumslag-Solitar group and $\theta : (G, P) \to (\mathbb{Z}, \mathbb{N})$ the height map. Then

$$\theta^{-1}(q) \cap P = \{b^{s_0}ab^{s_1} \cdots b^{s_{q-1}}ab^{s_n} : n \geq 0\} \quad \text{and}$$
$$\Sigma_q = \{b^{s_0}ab^{s_1} \cdots b^{s_{q-1}}a\}.$$

Then θ is a controlled map with amenable kernel.

- Let F be the Thompsen group and $\phi : (F, P) \to (\mathbb{Z}, \mathbb{N})$ be the homomorphism such that $\phi(x_0) = 1$ and $\phi(x_i) = 0$ for $i \geq 1$. Then $\Sigma_q = \phi^{-1}(q) \cap P = \{x_0^q\}$, and ϕ is a controlled map with kernel isomorphic to (F, P).

Theorem (aH-Tolich-Raeburn)

We found conditions on an HNN-extension (G^*, P^*) of a quasi-lattice ordered group (G, P) such that

1. (G^*, P^*) is quasi-lattice ordered, and
2. if (G, P) is amenable, then so is (G^*, P^*).

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