

Amenability of quasi-lattice ordered groups

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Overview

- 1 Quasi-lattice ordered groups: definition and examples
- 2 Isometries
- 3 C^* -algebras of quasi-lattice ordered groups
- 4 Amenability of quasi-lattice ordered groups

Let P be a generating submonoid of a discrete group G st $P \cap P^{-1} = \{e\}$. For $x, y \in G$ define

$$x \leq y \iff x^{-1}y \in P \iff y \in xP.$$

Then \leq is a partial order on G .

Definition (Nica, 1992)

(G, P) is *quasi-lattice ordered* if all $x, y \in G$ with a common upper bound in P have a least common upper bound $x \vee y$ in P .

Equivalently,

- 1 Any $x \in PP^{-1}$ has a least upper bound in P ; and
- 2 any $x, y \in P$ with a common upper bound have a least common upper bound.

(Item (1) implies (2).)

Write $x \vee y < \infty$ if the least upper bound exists, $x \vee y = \infty$ else.

From above: (G, P) is *quasi-lattice ordered* if all $x, y \in G$ with a common upper bound in P have a least common upper bound $x \vee y$ in P .

Examples

- 1 In $(\mathbb{Z}^2, \mathbb{N}^2)$, have $(m_1, m_2) \leq (n_1, n_2)$ iff $m_i \leq n_i$. So $(m_1, m_2) \vee (n_1, n_2) = (\max(m_1, n_1, 0), \max(m_2, n_2, 0))$.
- 2 $(\mathbb{Q}_+^*, \mathbb{N}^\times)$ under multiplication. For $m, n \in \mathbb{N}^\times$, $m \leq n$ means $m \mid n$, and $m \vee n$ is the lowest common multiple.
- 3 Let G be the free group with generators a, b , and P the submonoid consisting of words in a and b . Then $x \leq y$ means that x is an initial segment of y , and the rest of y has no factors of a^{-1} or b^{-1} . Here $x \vee y = \infty$ often.

Motivating example

Let $c, d \in \mathbb{N}^+$. The *Baumslag-Solitar group* has presentation

$$G = \langle a, b : ab^c = b^d a \rangle.$$

Let P be the submonoid generated by a and b . Then (G, P) is quasi-lattice ordered (Spielberg, 2012). Crucial is that G is an *HNN extension* of \mathbb{Z} , and hence each $x \in G$ has a unique *normal form*. Write $\theta : G \rightarrow \mathbb{Z}$ for the homomorphism such that $\theta(a) = 1$ and $\theta(b) = 0$. If $x \in P$, then

$$x = b^{s_0} a b^{s_1} \dots b^{s_{k-1}} a b^{s_k}.$$

where $k = \theta(x)$ and each $0 \leq s_{i-1} < d$. We call θ the *height map*.

- $a^2 b^{42} = b^0 a b^0 a b^{42}$
- $b^d a = a b^c = b^0 a b^c$
- If $0 \leq n < d$, then $b^{n+d} a = b^n a b^c$.

Spielberg's proof that the B-S group is quasi-lattice ordered uses that (G, P) is quasi-lattice ordered iff (Crisp-Laca, 2002)

if $x \in PP^{-1}$, then there exist a pair $\mu, \nu \in P$ with $x = \mu\nu^{-1}$ such that $\gamma, \delta \in P$ and $\gamma\delta^{-1} = \mu\nu^{-1}$ imply $\mu \leq \gamma$ and $\nu \leq \delta$. (The pair μ, ν is unique.)

Tease: Thompson's group

$$F = \langle \{x_i : i \in \mathbb{N}\} : x_i x_k = x_k x_{i+1} \text{ for } i > k \rangle.$$

Let P be the submonoid generated by $\{x_i : i \in \mathbb{N}\}$. Then (F, P) is quasi-lattice ordered (Nucinkis, 2017). In fact, (F, P) is lattice ordered.

Let H be a Hilbert space. A linear bounded operator S on H is an *isometry* if $\|Sh\| = \|h\|$ for all $h \in H$. Equivalently, $S^*S = 1$.

Example

$H = \ell^2(\mathbb{N})$, the square-summable sequences. The unilateral shift S

$$S(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$$

is an isometry with *adjoint* $S^*(x_0, x_1, \dots) = (x_1, x_2, \dots)$. The *range projection* SS^* is the orthogonal projection onto the subspace of sequences with a zero in the first entry.

An *isometric representation* of a monoid P is a homomorphism $S : P \rightarrow B(H)$ such that each S_p is an isometry.

Example

Consider $\ell^2(P)$ with o.n. basis $\{e_x : x \in P\}$. For each $x \in P$, there is an isometry T_x on $\ell^2(P)$ such that $T_x e_y = e_{xy}$ for $y \in P$. Then T is an isometric representation of P .

Suppose that (G, P) is a quasi-lattice ordered group. Let $T : P \rightarrow \ell^2(P)$ be $T_x e_y = e_{xy}$ for $y \in P$ as above. T is called the *Toeplitz representation*. Nica observed that

$$T_x T_x^* T_y T_y^* = \begin{cases} T_{(x \vee y)} T_{(x \vee y)}^* & \text{if } x \vee y < \infty \\ 0 & \text{if } x \vee y = \infty. \end{cases} \quad (1)$$

Equivalently, $T_x^* T_y = \begin{cases} T_{x^{-1}(x \vee y)} T_{y^{-1}(x \vee y)}^* & \text{if } x \vee y < \infty \\ 0 & \text{if } x \vee y = \infty. \end{cases}$

An isometric rep of P satisfying (1) is *Nica-covariant*.

(G, P) has two C^* -algebras:

- 1 The *Toeplitz algebra* is $\mathcal{T}(P) := \langle T_x : x \in P \rangle \subset B(\ell^2(P))$.
- 2 The C^* -algebra $C^*(G, P) = \overline{\text{span}}\{w_x w_y^* : x, y \in P\}$ universal for Nica-covariant reps. (Here w is a universal Nica-covariant rep.)

The Toeplitz representation $T : P \rightarrow \mathcal{T}(P)$ induces a surjection $\pi_T : C^*(G, P) \rightarrow \mathcal{T}(P)$.

Example: Baumslag-Solitar $G = \langle a, b : ab^c = b^d a \rangle$.

Thm. (Clark-aH-Raeburn, 2016)

$C^*(G, P)$ is universal for isometries U and V satisfying

- 1 $VU^c = U^d V$;
- 2 $U^* V = U^{d-1} VU^{*c}$;
- 3 $V^* U^j V = 0$ for $1 \leq j < d$.

It's easy to see that the Toeplitz representation satisfies (1)-(3) with $U := T_b$ and $V := T_a$ in $B(l^2(P))$.

- 1 $VU^c = T_a T_b^c = T_{ab^c} = T_{b^d a} = T_{b^d} T_a = U^d V$.
- 2 Since $a \vee b = ab^c = b^d a$, Nica covariance gives

$$U^* V = T_b^* T_a = T_{b^{-1}(a \vee b)} T_{a^{-1}(a \vee b)}^* = T_{b^{d-1} a} T_{b^c}^* = U^{d-1} VU^{*c}.$$

- 3 Let $1 \leq j < d$. Then $a \vee b^j a = \infty$, Nica covariance says $V^* U^j V = 0$.

Say (G, P) is *amenable* if the Toeplitz rep induces an isomorphism of the universal C^* -algebra $C^*(G, P)$ onto the Toeplitz C^* -algebra.

- There exists a linear idempotent $E : C^*(G, P) \rightarrow \overline{\text{span}}\{w_p w_p^* : p \in P\}$. Then (G, P) is amenable if and only if $E(a^* a) = 0$ implies $a = 0$.
- There are Følner-type conditions for amenability.

Tease:

Thompson's group F is amenable if and only if (F, P) is amenable as a quasi-lattice ordered group.

Examples of amenable (G, P)

- (Nica) If G is an amenable group, then (G, P) is an amenable quasi-lattice ordered group.
- (Nica) Let \mathbb{F}_2 be the free group on $\{a, b\}$ and let $\mathbb{F}_2^+ = \langle a, b \rangle$. Then $(\mathbb{F}_2, \mathbb{F}_2^+)$ is amenable.
- (Clark-aH-Raeburn, 2016) The Baumslag-Solitar (G, P) is amenable. To prove this we had to generalise the proof of:

Theorem (Laca-Raeburn, 1996)

Let (G, P) and (K, Q) be quasi-lattice ordered groups and let $\mu : G \rightarrow K$ be “controlled”. If K is amenable, then (G, P) is.

Controlled means: μ is an order-preserving group homomorphism such that for $x, y \in P$ with $x \vee y < \infty$:

① $\mu(x) \vee \mu(y) = \mu(x \vee y)$;

② $\mu(x) = \mu(y) \Rightarrow x = y$.

Note: $\mu^{-1}(e) \cap P = \{e\}$.

The Baumslag-Solitar height map θ is not controlled since $\theta^{-1}(e) \cap P = \{b^n : n \in \mathbb{N}\}$.

We used ideas from Clark-aH-Raeburn about the B-S group to find a new notion of “controlled map” and then used the strategy from Laca-Raeburn to show amenability.

Theorem (aH-Raeburn-Tolich, preprint 2017)

Suppose that $\mu : (G, P) \rightarrow (K, Q)$ is controlled (in a new technical sense).

- 1 $(\mu^{-1}(e), \mu^{-1}(e) \cap P)$ is a quasi-lattice ordered group.
- 2 If K is an amenable group and $(\mu^{-1}(e), \mu^{-1}(e) \cap P)$ is an amenable quasi-lattice group, then (G, P) is amenable.

The idea for this comes from:

Theorem (Spielberg, 2014)

Let G be a Hausdorff étale groupoid, let Q be a countable abelian group, and let $c : G \rightarrow Q$ be a continuous homomorphism. Then

- 1 $c^{-1}(0)$ is a Hausdorff étale groupoid.
- 2 If $c^{-1}(0)$ is an amenable groupoid, then G is.

The new notion of “controlled” map

Let $\mu : G \rightarrow K$ be an order-preserving group homomorphism. Then μ is *controlled* if it has the following properties:

- 1 For $x, y \in P$ with $x \vee y < \infty$, we have $\mu(x) \vee \mu(y) = \mu(x \vee y)$.
- 2 Let $q \in Q$. Set $\Sigma_q = \{\sigma \in \mu^{-1}(q) \cap P : \sigma \text{ is minimal}\}$.
 - a. If $x \in \mu^{-1}(q) \cap P$, then there exists $\sigma \in \Sigma_q$ such that $\sigma \leq x$.
 - b. If $\sigma, \tau \in \Sigma_q$, then $\sigma \vee \tau < \infty \Rightarrow \sigma = \tau$.

Examples

- Let G be the Baumslag-Solitar group and $\theta : (G, P) \rightarrow (\mathbb{Z}, \mathbb{N})$ the height map. Then

$$\theta^{-1}(q) \cap P = \{b^{s_0} a b^{s_1} \dots b^{s_{q-1}} a b^{s_n} : n \geq 0\} \quad \text{and}$$
$$\Sigma_q = \{b^{s_0} a b^{s_1} \dots b^{s_{q-1}} a\}.$$








Then θ is a controlled map with amenable kernel.

- Let F be the Thompson group and $\phi : (F, P) \rightarrow (\mathbb{Z}, \mathbb{N})$ be the homomorphism such that $\phi(x_0) = 1$ and $\phi(x_i) = 0$ for $i \geq 1$. Then $\Sigma_q = \phi^{-1}(q) \cap P = \{x_0^q\}$, and ϕ is a controlled map with kernel isomorphic to (F, P) .

Theorem (aH-Tolich-Raeburn)

We found conditions on an HNN-extension (G^*, P^*) of a quasi-lattice ordered group (G, P) such that

- 1 (G^*, P^*) is quasi-lattice ordered, and
- 2 if (G, P) is amenable, then so is (G^*, P^*) .

-  L.O. Clark, A. an Huef, I. Raeburn, *Phase transitions on the Toeplitz algebras of Baumslag-Solitar semigroups*, Indiana Univ. Math. J. (2016).
-  J. Crisp, M. Laca, *On the Toeplitz algebras of right-angled and finite-type Artin groups*, J. Austral. Math. Soc. (2002).
-  A. an Huef, I. Raeburn, I. Tolich, *HNN extensions of quasi-lattice ordered groups and their operator algebras*.
-  M. Laca, I. Raeburn, *Semigroup crossed products and the Toeplitz algebras of nonabelian groups*, J. Funct. Anal. (1996).
-  A. Nica, *C^* -algebras generated by isometries and Wiener-Hopf operators*, J. Operator Theory (1992).
-  J. Spielberg, *C^* -algebras for categories of paths associated to the Baumslag-Solitar groups*, J. London Math. Soc. (2012).
-  J. Spielberg, *Groupoids and C^* -algebras for categories of paths*, Trans. Amer. Math. Soc. (2014).