Self-similar groupoid actions

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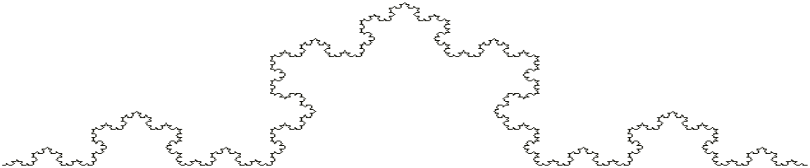
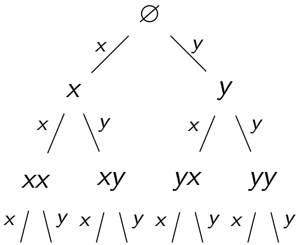


Trees and automorphisms

- Suppose X is a finite set;
  - let X^k denote the set of words of length k in X with $X^0 = \varnothing$,
  - let $X^* = \bigcup_{k \geq 0} X^k$.
- Consider the rooted tree $T = T_X$ with vertex set $T^0 = X^*$ and edge set $T^1 = \{\{\mu, \mu x\} : \mu \in X^* \text{ and } x \in X\}$.

T_X for $X = \{x, y\}$



- An automorphism of T is a bijection $a : T^0 \to T^0$ such that
  - $a(X^k) = X^k$ for all k and
  - $a(\mu x) \in a(\mu)X$ for all $\mu \in X^k$ and $x \in X$.

Automata on X

Definition: An automaton over X is a finite set A together with a map

$$\begin{array}{rcl} A \times X & \to & X \times A \\ (a, x) & \mapsto & (a \cdot x, a|_x) \end{array}$$

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- For $\mu \in X^k$ and $a \in A$ we define $a|_\mu$ inductively by

$$a|_{\mu x} = (a|_\mu)|_x.$$

For $k \geq 0$ and fixed $a \in A$, we extend $a : X^k \to X^k$ by

$$a \cdot \mu = (a \cdot \mu_1)(a|_{\mu_1} \cdot \mu_2) \cdots (a|_{\mu_1 \cdots \mu_{k-1}} \cdot \mu_k)$$

for $\mu = \mu_1 \cdots \mu_k \in X^k$. In this way, each $a \in A$ extends to an automorphism of T_X.

Self-similar groups

Definition: Suppose X is a finite set and G is a group acting faithfully on T_X. We say (G, X) is a self-similar group if, for all $g \in G$ and $x \in X$, there exist $h \in G$ such that

$$g \cdot (xw) = (g \cdot x)(h \cdot w) \quad \text{for all finite words } w \in X^*. \tag{1}$$

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Proposition: Suppose A is an automaton over X. Then the subgroup G_A of $\mathrm{Aut}(T_X)$ generated by A is a self-similar group.

Example: the odometer

- Let $X = \{0, 1\}$ and let A be the automaton defined by

$$\begin{aligned} a \cdot 0 &= 1 & \qquad a \cdot 1 &= 0 \\ a|_0 &= e & \qquad a|_1 &= a \end{aligned}$$

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- To see how elements of automata extend to automorphisms we compute the extension of $a \in \mathrm{Aut}(T_X)$ on the word 11101:

$$\begin{aligned} a \cdot 11101 &= (a \cdot 1)(a|_1 \cdot 1)(a|_1|_1 \cdot 1)(a|_1|_1|_1 \cdot 0)(a|_1|_1|_1|_0 \cdot 1) \\ &= (a \cdot 1)(a \cdot 1)(a \cdot 1)(a \cdot 0)(e \cdot 1) \\ &= 00011 \end{aligned}$$

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- The group G_A generated by A is the integers $\mathbb{Z} := \{a^n : n \in \mathbb{Z}\}$. The self similar group (G_A, X) is commonly called the odometer because the self-similar action is "adding one with carryover in binary."

Example: the Grigorchuk group

- Suppose $X = \{x, y\}$ and let $A = \{a, b, c, d, e\}$ be the automaton defined by

$$
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a \cdot x = y & a|_x = e & a \cdot y = x & a|_y = e \\
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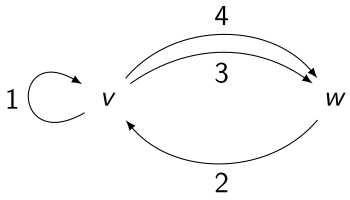
Theorem (Grigorchuk 1982): The Grigorchuk group has intermediate growth.

Directed graphs

- Let $E = (E^0, E^1, r, s)$ be a finite directed graph with vertex set E^0, edge set E^1, and range and source maps from E^1 to E^0.



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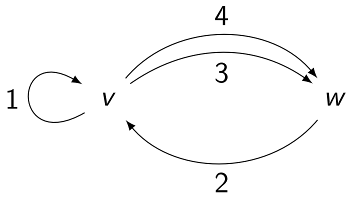

- Given a graph E, the set of paths of length k is

$$E^k := \{\mu = \mu_1\mu_2\cdots\mu_k : \ \mu_i \in E^1, s(\mu_i) = r(\mu_{i+1})\},$$

and let

$$E^* = \bigcup_{k=0}^{\infty} E^k$$

denote the collection of finite paths. A path of length zero is defined to be a vertex.

Trees and forests

- The analogue of the tree T_X is the undirected graph T_E with vertex set E^* and edge set

$$T^1 = \left\{ \{\mu, \mu e\} : \ \mu \in E^*, e \in E^1, \text{ and } s(\mu) = r(e) \right\}.$$

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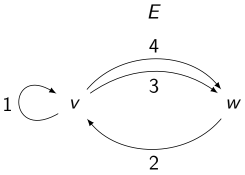
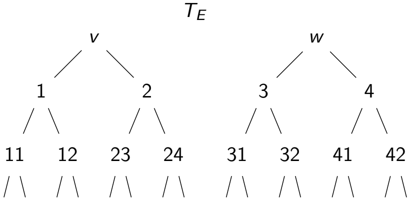
The subgraph $vE^* = \{\mu \in E^* : r(\mu) = v\}$ is a rooted tree, and $T_E = \bigsqcup_{v \in E^0} vE^*$ is a disjoint union of trees.





Partial isomorphisms on graphs

- A *partial isomorphism* of T_E consists of two vertices $v, w \in E^0$ and a bijection $g : vE^* \to wE^*$ such that
  - $g|_{vE^k}$ is a bijection onto wE^k for $k \in \mathbb{N}$, and
  - $g(\mu e) \in g(\mu)E^1$ for all $\mu e \in vE^*$.

  For $v \in E^0$, we write $\mathrm{id}_v : vE^* \to vE^*$ for the identity partial isomorphism.

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- We denote the set of all partial isomorphisms of T_E by $\mathcal{P}(E^*)$
- Define domain and codomain maps $d, c \colon \mathcal{P}(E^*) \to E^0$ so that $g : d(g)E^* \to c(g)E^*$.

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Proposition: The set $\mathcal{P}(E^*)$ of partial isomorphisms on T_E is a groupoid with unit space E^0. In particular, we have

- if $d(g) = c(h)$, the product $gh : d(h)E^* \to c(g)E^*$ is defined by composition, and
- $g^{-1} : c(g)E^* \to d(g)E^*$ is the inverse of g.

Graph automata

Definition: An automaton over E is a finite set A containing E^0 together with functions $c, d\colon A \to E^0$ and a function

$$A \,{}_d\!\times_r E^1 \ni (a, e) \mapsto (a \cdot e, a|_e) \in E^1 \,{}_s\!\times_c A$$

such that:

(A1) for each $a \in A$, $e \mapsto a \cdot e$ is a bijection of $d(a)E^1$ onto $c(a)E^1$;

(A2) $d(a|_e) = s(e)$ for all $(a, e) \in A \,{}_d\!\times_r E^1$;

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- As before, each $a \in A$ extends to a map $a : d(a)E^* \to c(a)E^*$ by the formula

  $$a \cdot \mu = (a \cdot \mu_1)(a|_{\mu_1} \cdot \mu_2) \cdots (a|_{\mu_1 \cdots \mu_{k-1}} \cdot \mu_k) \text{ for } \mu \in s(a)E^*$$

  So each $a \in A$ becomes a partial isomorphism of T_E.

Self-similar groupoid actions

- An action of G on T_E is a (unit-preserving) groupoid homomorphism $\phi : G \to \mathcal{P}(E^*)$, and the action is faithful if ϕ is one-to-one.
- We usually write $g \cdot \mu$ for $\phi_g(\mu)$.

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Definition Suppose E is a directed graph and G is a groupoid with unit space E^0 acting faithfully on T_E. Then (G, E) is a self-similar groupoid action if, for every $g \in G$ and $e \in d(g)E^1$, there exists $h \in G$ satisfying

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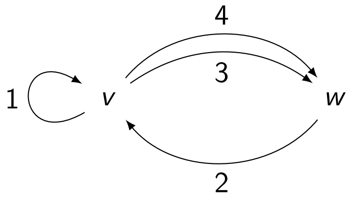
- Let E be the graph



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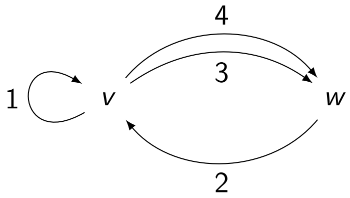
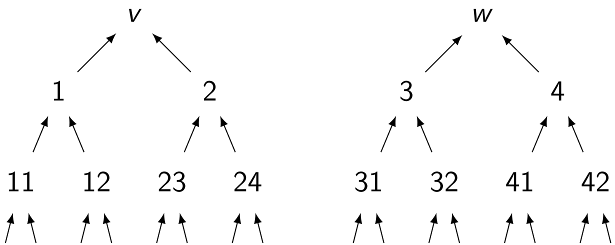


- The path space E^* is



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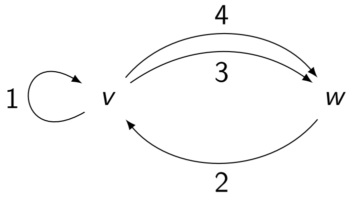

- Let $A = \{v, w, a, b\}$, where a and b are defined by

$$\begin{array}{llll} a \cdot 1 = 4, & a|_1 = v; & a \cdot 2 = 3, & a|_2 = b; \\ b \cdot 3 = 1, & b|_3 = v; & b \cdot 4 = 2, & b|_4 = a. \end{array}$$

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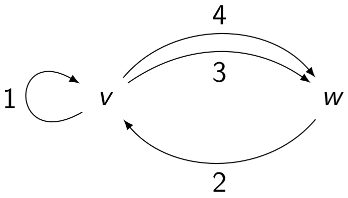

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- One can check that this satisfies all the definitions of an E-automaton, and hence gives rise to a self-similar groupoid G_A acting on T_E.

References

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