On base radical and semisimple class operators for associative rings

Lauren Thornton
in collaboration with Dr Robert McDougall

University of the Sunshine Coast

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I would like to acknowledge and thank the Australian Algebra Group, as well as the University of the Sunshine Coast, for providing funding to attend and present at the 1^{st} Australian Algebra Conference, 2017.

Background Preliminaries

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- $\mathcal{X}$ is *closed under extensions* if $\mathcal{X}$ has the property that whenever $I \triangleleft A \in \mathcal{A}$ and both I and A/I are elements of $\mathcal{X}$, then A is in $\mathcal{X}$

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$\mathbf{U}(\mathcal{X}) = \{A \in \mathcal{A} \mid A \text{ has no nonzero homomorphic image in } \mathcal{X}\}$

$\mathbf{S}(\mathcal{X}) = \{A \in \mathcal{A} \mid A \text{ has no nonzero accessible subring in } \mathcal{X}\}$

Background Preliminaries

- For the trivial classes in $\mathcal{A}$ we have $\mathbf{U}(0) = \mathbf{S}(0) = \mathcal{A}$, and $\mathbf{U}(\mathcal{A}) = \mathbf{S}(\mathcal{A}) = 0$

- For all subclasses $\mathcal{X} \subseteq \mathcal{A}$ we have $\mathcal{X} \cap \mathbf{U}(\mathcal{X}) = \mathcal{X} \cap \mathbf{S}(\mathcal{X}) = 0$

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Meaningful outcomes generated by interchanging the operators **U** and **S** are called *dual* results.

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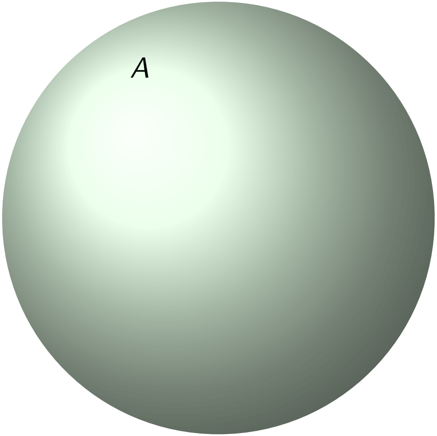
(R1) If $A \in \mathcal{R}$, then every nonzero homomorphic image of A has a nonzero ideal in $\mathcal{R}$;

(R2) If every nonzero homomorphic image of a ring A has a nonzero ideal in $\mathcal{R}$, then $A \in \mathcal{R}$.

Background Preliminaries



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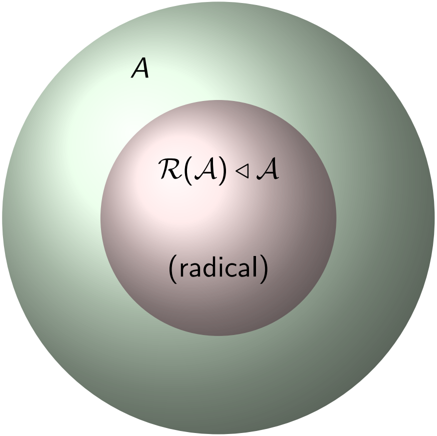

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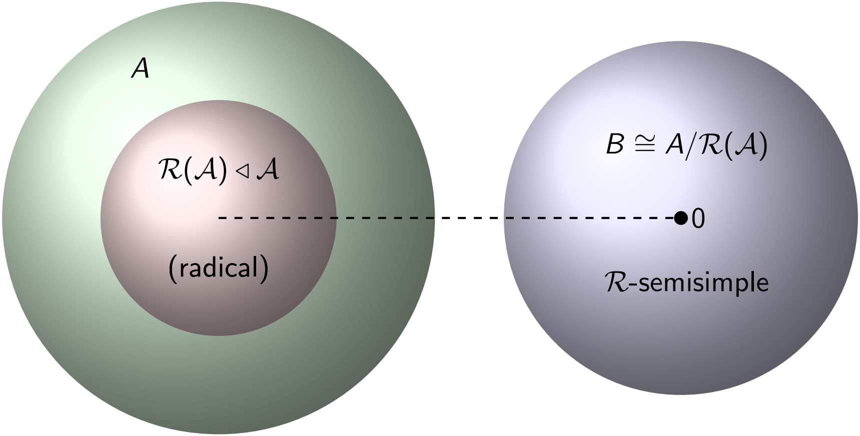

Background Preliminaries

The *base radical operator* **US** and *base semisimple operator* **SU**[1] are then

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The *base radical operator* **US** and *base semisimple operator* **SU**[1] are then

$\mathbf{US}(\mathcal{X}) = \{A \in \mathcal{A} \,|\,$every nonzero homomorphic image of A has a nonzero accessible subring in $\mathcal{X}\}$

$\mathbf{SU}(\mathcal{X}) = \{A \in \mathcal{A} \,|\,$every nonzero accessible subring of A has a nonzero homomorphic image in $\mathcal{X}\}$

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Background Preliminaries

- A class $\mathcal{X} \subseteq \mathcal{A}$ is a *base radical class* iff $\mathcal{X} = \mathbf{US}(\mathcal{X})$, and dually
- $\mathcal{X}$ is a *base semisimple class* iff $\mathcal{X} = \mathbf{SU}(\mathcal{X})$ [2]

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For associative rings, Kurosh-Amitsur radical classes are precisely base radical classes and, since Kurosh-Amitsur semisimple classes are hereditary, their corresponding semisimple classes coincide.

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- A class $\mathcal{X}$ is called *radical-semisimple* if $\mathcal{X}$ is both a radical class and a semisimple class

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Background Preliminaries

The operator semigroup $RT_{\mathcal{A}}$

- contains all distinct operators composed of **U** and **S** that can act on subclasses of a universal class $\mathcal{A}$

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Previously described for the universal class of two and four element cyclic groups and for the universal class of all simple associative rings. [3]

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- For example, the universal class generated by associative ring $\mathbb{Z}_4$ ($\mathcal{A}_1 = \{0, \mathbb{Z}_4, \mathbb{Z}_2, \mathbb{Z}_2^0\}$) has associated operator semigroup

  $RT_{\mathcal{A}_1} = \{\mathbf{U}, \mathbf{S}, \mathbf{UU}, \mathbf{US}, \mathbf{SU}, \mathbf{SS}, \mathbf{UUU}, \mathbf{USS}, \mathbf{USU}, \mathbf{SUS}, \mathbf{SSS}, \mathbf{USSS}\}$ with $|RT_{\mathcal{A}_1}| = 12$

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For all subclasses $\mathcal{X} \subseteq \mathcal{A}$,

- $\neg(\mathcal{X}) = \{A \in \mathcal{A} \mid A \notin \mathcal{X}\}$

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- $\neg(\mathcal{X}) = \{A \in \mathcal{A} \mid A \notin \mathcal{X}\}$
- Denote the operator semigroup formed by operators $\mathbf{U}, \mathbf{S}$ and $\neg$ by $RT^*_{\mathcal{A}}$

On Base Radical and Semisimple Operators

Proposition

If a subclass $\mathcal{X} \subseteq \mathcal{A}$ *is closed under extensions, then* $\mathbf{SU}(\mathcal{X}) = \mathbf{S}\neg(\mathcal{X})$.

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Proof.

Let a subclass $\mathcal{X}$ of $\mathcal{A}$ be closed under extensions with $A \in \mathbf{SU}(\mathcal{X})$ such that $A \notin \mathcal{X}$, and C_1 a nonzero accessible subring of A. Suppose $C_1 \notin \mathcal{X}$. $C_1 \in \mathbf{SU}(\mathcal{X})$ since $\mathbf{SU}(\mathcal{X})$ is hereditary, hence C_1 has a nonzero homomorphic image $D_1 \in \mathcal{X}$ such that $D_1 \cong C_1 / C_2$ for some nonzero ideal C_2 of C_1.

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On Base Radical and Semisimple Operators

Necessary and sufficient conditions for a class of associative rings to be semisimple are that the class be hereditary, closed under subdirect sums and closed under extensions.[4]

[4]A. D. Sands, 'Strong upper radicals', *Quart. J. Math. Oxford* **27** (1976), 21–24

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Proposition

For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\mathcal{X}$ is hereditary and closed under extensions iff $\mathcal{X}$ is a semisimple class.

Proof.

If $\mathcal{X}$ is hereditary and closed under extensions then $\mathbf{SU}(\mathcal{X}) = \mathbf{S}\neg(\mathcal{X}) = \mathcal{X}$ and so $\mathcal{X}$ is semisimple. The converse is clear. □

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Theorem

For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{SUU}(\mathcal{X})$ is a radical-semisimple class.

Proof.

Since $\mathbf{UUU}(\mathcal{X})$ is a radical class for all $\mathcal{X} \subseteq \mathcal{A}$ and hence closed under extensions. The first proposition gives $\mathbf{SU}(\mathbf{UUU}(\mathcal{X})) = \mathbf{S}\neg(\mathbf{UUU}(\mathcal{X}))$ and so $\mathbf{SUU}(\mathcal{X}) = \mathbf{S}\neg\mathbf{UUU}(\mathcal{X})$. The class $\mathbf{S}\neg\mathbf{UUU}(\mathcal{X})$ is homomorphically closed and since any homomorphically closed semisimple class is a radical class [5], $\mathbf{SUU}(\mathcal{X})$ is a radical-semisimple class. □

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Proposition

For all subclasses $\mathcal{X} \subseteq \mathcal{A}, \mathbf{USS}(\mathcal{X}) = \mathbf{SUU}(\mathcal{X})$.

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Consider two properties a class $\mathcal{X} \subseteq \mathcal{A}$ may have:

(1*) If $A \in \mathcal{X}$ then A has a simple nonzero homomorphic image in $\mathcal{X}$,

(2*) If $A \in \mathcal{X}$ then A has a simple nonzero accessible subring in $\mathcal{X}$.

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Proposition

For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, the following are equivalent:

(i) $\mathcal{X} = \mathbf{US}(\mathcal{X}) = \mathbf{SU}(\mathcal{X})$,

(ii) $\mathcal{X} = \mathbf{USSS}(\mathcal{X}) = \mathbf{SUUU}(\mathcal{X})$,

(iii) $\mathcal{X}$ *is a radical class with condition (2*),*

(iv) $\mathcal{X}$ *is a semisimple class with condition (1*).*

On Base Radical and Semisimple Operators

Proposition

A radical class $\mathcal{R}$ is hereditary iff $\mathcal{R}$ is of the form $\mathbf{USS}(\mathcal{X})$ *for some subclass $\mathcal{X}$ of $\mathcal{A}$.*

Proof.

If a radical class $\mathcal{R}$ is hereditary then $\mathcal{R}$ has condition (2^*) and, by the previous proposition, $\mathcal{R} = \mathbf{USSS}(\mathcal{R})$ and so $\mathcal{R}$ is of the form $\mathbf{USS}(\mathcal{X})$. The converse is clear from the previous proposition. □

On Base Radical and Semisimple Operators

For the universal class of all associative rings the 'concrete radical classes', including the Jacobson radical, are all hereditary[6] and it follows that these are all radical-semisimple classes in this setting.

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Proposition

If condition (2^*) *holds for a subclass* $\mathcal{X} \subseteq \mathcal{A}$ *then the radical class* $\mathbf{US}(\mathcal{X})$ *is a radical-semisimple class. Dually, if condition* (1^*) *holds for a subclass* $\mathcal{X} \subseteq \mathcal{A}$ *then the semisimple class* $\mathbf{SU}(\mathcal{X})$ *is a radical-semisimple class.*

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It is known that the largest hereditary subclass of a radical class is a radical class[7], and in this finite setting we can extend this to

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The largest hereditary subclass of a radical class is a radical-semisimple class. Dually, the largest homomorphically closed subclass of a semisimple class is a radical-semisimple class.

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For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{SUUU}(\mathcal{X}) = \mathbf{S}\neg\mathbf{USSS}(\mathcal{X}) = \mathbf{S}\neg\mathbf{USU}\neg(\mathcal{X}) = \mathbf{S}\neg\mathbf{US}(\mathcal{X}) = \mathbf{S}\neg\mathbf{UUU}(\mathcal{X}) = \mathbf{U}\neg\mathbf{SUUU}(\mathcal{X}) = \mathbf{U}\neg\mathbf{SUS}\neg(\mathcal{X}) = \mathbf{U}\neg\mathbf{SU}(\mathcal{X}) = \mathbf{USSS}(\mathcal{X})$.

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For any subclass $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{S}(\mathcal{X})$ is a semisimple class when $\mathcal{X}$ is homomorphically closed[8] or hereditary.

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Proposition

For a subclass $\mathcal{X} \subseteq \mathcal{A}$, if $\mathcal{X} \subseteq \mathbf{SU}(\mathcal{X})$ then $\mathbf{S}(\mathcal{X})$ is a semisimple class.

Proof.

If a subclass $\mathcal{X}$ of $\mathcal{A}$ is contained in $\mathbf{SU}(\mathcal{X})$, then $\mathbf{SSU}(\mathcal{X}) \subseteq \mathbf{S}(\mathcal{X}) \subseteq \mathbf{SSS}(\mathcal{X})$. Since $\mathbf{SSU}(\mathcal{X}) = \mathbf{SSS}(\mathcal{X})$ it follows that $\mathbf{S}(\mathcal{X}) = \mathbf{SSS}(\mathcal{X})$, showing $\mathbf{S}(\mathcal{X})$ is a semisimple class. □

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On Base Radical and Semisimple Operators

The correspondence theorem shows that, for all $A \in \mathcal{A}$, every accessible subring of a homomorphic image of A is a homomorphic image of an accessible subring of A. We can dualise this result to an extent for finite associative rings as follows.

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Proposition

For all $A \in \mathcal{A}$, every simple homomorphic image of an accessible subring of A is a simple accessible subring of a homomorphic image of A.

Proof.

Note that since $\mathbf{USSS}(\mathcal{X}) = \mathbf{SUUU}(\mathcal{X})$ for all $\mathcal{X} \subseteq \mathcal{A}$ we have $\mathbf{USSS}(\mathcal{X}) = \mathbf{SUUU}(\mathcal{X}) = \mathbf{S}\neg\mathbf{UU}(\mathcal{X}) \subseteq \mathbf{U}\neg\mathbf{SU}(\mathcal{X}) \subseteq \mathbf{U}\neg\mathbf{SS}(\mathcal{X})$ $\subseteq \mathbf{USSS}(\mathcal{X})$ and so all classes are equal. Suppose C is a simple homomorphic image of an accessible subring of $A \in \mathcal{A}$ such that C is not the simple accessible subring of a homomorphic image of A. Then $A \in \neg\mathbf{S}\neg\mathbf{UUU}(\{C\})$ but $A \notin \neg\mathbf{U}\neg\mathbf{SSS}(\{C\})$. Hence $\mathbf{S}\neg\mathbf{UU}(\mathcal{X}) \neq \mathbf{U}\neg\mathbf{SS}(\mathcal{X})$, a contradiction. □

On Base Radical and Semisimple Operators

For any subclass $\mathcal{X} \subseteq \mathcal{A}$,

$\mathcal{X}_s = \{A \in \mathcal{A} \mid A \in \mathcal{X} \text{ and } A \text{ is simple}\}$

$\mathcal{X}_s$ is hereditary and homomorpically closed and generates coinciding lower radical class and semisimple closure.

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Proposition

For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{US}(\mathcal{X}_s) = \mathbf{SU}(\mathcal{X}_s)$.

On Base Radical and Semisimple Operators

We now determine the maximum order of $RT_{\mathcal{A}}$ and $RT^*_{\mathcal{A}}$ for universal classes of finite associate rings, including a complete listing of possible radical and semisimple class operators.

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Theorem

For the operator semigroup $RT_{\mathcal{A}}$, $|RT_{\mathcal{A}}| \leq 12$.

Proof.

Starting with the generating classes $\mathbf{U}(\mathcal{X})$ and $\mathbf{S}(\mathcal{X})$ and applying $\mathbf{U}$ and $\mathbf{S}$ to each in turn we have $\mathbf{UU}(\mathcal{X})$, $\mathbf{US}(\mathcal{X})$, $\mathbf{SU}(\mathcal{X})$ and $\mathbf{SS}(\mathcal{X})$. Repeating generates classes $\mathbf{UUU}(\mathcal{X})$, $\mathbf{SUU}(\mathcal{X})$, $\mathbf{USU}(\mathcal{X})$, $\mathbf{SUS}(\mathcal{X})$ and $\mathbf{SSS}(\mathcal{X})$, since other possibilites collapse. A further application of $\mathbf{S}$ gives the last distinct class $\mathbf{SUUU}(\mathcal{X})$. In summary
$RT_{\mathcal{A}} = \{\mathbf{U}, \mathbf{S}, \mathbf{UU}, \mathbf{US}, \mathbf{SU}, \mathbf{SS}, \mathbf{UUU}, \mathbf{SUU}, \mathbf{USU}, \mathbf{SUS}, \mathbf{SSS}, \mathbf{SUUU}\}$
showing that the order of $RT_{\mathcal{A}}$ is at most 12. □

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For the operator semigroup $RT_{\mathcal{A}}$, $|RT_{\mathcal{A}}| \leq 12$.

Proof.

Starting with the generating classes $\mathbf{U}(\mathcal{X})$ and $\mathbf{S}(\mathcal{X})$ and applying $\mathbf{U}$ and $\mathbf{S}$ to each in turn we have $\mathbf{UU}(\mathcal{X})$, $\mathbf{US}(\mathcal{X})$, $\mathbf{SU}(\mathcal{X})$ and $\mathbf{SS}(\mathcal{X})$. Repeating generates classes $\mathbf{UUU}(\mathcal{X})$, $\mathbf{SUU}(\mathcal{X})$, $\mathbf{USU}(\mathcal{X})$, $\mathbf{SUS}(\mathcal{X})$ and $\mathbf{SSS}(\mathcal{X})$, since other possibilites collapse. A further application of $\mathbf{S}$ gives the last distinct class $\mathbf{SUUU}(\mathcal{X})$. In summary
$RT_{\mathcal{A}} = \{\mathbf{U}, \mathbf{S}, \mathbf{UU}, \mathbf{US}, \mathbf{SU}, \mathbf{SS}, \mathbf{UUU}, \mathbf{SUU}, \mathbf{USU}, \mathbf{SUS}, \mathbf{SSS}, \mathbf{SUUU}\}$
showing that the order of $RT_{\mathcal{A}}$ is at most 12. □

This is the semigroup mentioned earlier for the universal class $\mathcal{A}_1$.

On Base Radical and Semisimple Operators

Using a similar proof strategy we can determine an upper bound for the order of the semigroup generated by $\mathbf{U}, \mathbf{S}$ and $\neg$.

On Base Radical and Semisimple Operators

Using a similar proof strategy we can determine an upper bound for the order of the semigroup generated by **U**, **S** and ¬.

Theorem

For the semigroup $RT^*_{\mathcal{A}}$, $|RT^*_{\mathcal{A}}| \leq 46$.

The complete list $RT^*_{\mathcal{A}} = \{$**U**, **S**, ¬, **UU**, **SU**, ¬**U**, **US**, **SS**, ¬**S**, **U**¬, **S**¬, ¬¬, **UUU**, **SUU**, ¬**UU**, **USU**, **SSS**, ¬**SU**, **S**¬**U**, **SUS**, ¬**US**, ¬**SS**, **U**¬**S**, **SU**¬, ¬**U**¬, **US**¬, ¬**S**¬, **SUUU**, ¬**UUU**, ¬**SUU**, ¬**USU**, ¬**SSS**, ¬**S**¬**U**, ¬**SUS**, ¬**U**¬**S**, **USU**¬, ¬**SU**¬, **S**¬**U**¬, **SUS**¬, ¬**US**¬, **U**¬**S**¬, ¬**SUUU**, ¬**USU**¬, ¬**S**¬**U**¬, ¬**SUS**¬, ¬**U**¬**S**¬$\}$.

On Base Radical and Semisimple Operators

Considering $\mathcal{A}_1 = \{0, \mathbb{Z}_4, \mathbb{Z}_2, \mathbb{Z}_2^0\}$, for all $A \in \mathcal{A}_1$, every homomorphic image of each accessible subring of A is an accessible subring of a homomorphic image of A, and so for all subclasses $\mathcal{X} \subseteq \mathcal{A}_1$ we have $\mathbf{S}\neg\mathbf{U}(\mathcal{X}) = \mathbf{U}\neg\mathbf{S}(\mathcal{X})$.

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Other equalities based on properties of the elements of $\mathcal{A}_1$ hold, and we have

$RT^*_{\mathcal{A}_1} = \{\mathbf{U}, \mathbf{S}, \neg, \mathbf{UU}, \mathbf{SU}, \neg\mathbf{U}, \mathbf{US}, \mathbf{SS}, \neg\mathbf{S}, \mathbf{U}\neg, \mathbf{S}\neg, \neg\neg, \mathbf{UUU}, \mathbf{SUU}, \mathbf{USU}, \mathbf{SSS}, \neg\mathbf{SU}, \mathbf{S}\neg\mathbf{U}, \mathbf{SUS}, \neg\mathbf{US}, \neg\mathbf{U}\neg, \neg\mathbf{S}\neg, \mathbf{SUUU}, \neg\mathbf{SUU}, \neg\mathbf{USU}, \neg\mathbf{S}\neg\mathbf{U}, \neg\mathbf{SUS}, \mathbf{S}\neg\mathbf{U}\neg, \neg\mathbf{SUUU}, \neg\mathbf{S}\neg\mathbf{U}\neg\}$

with $|RT^*_{\mathcal{A}_1}| = 30$.

On Base Radical and Semisimple Operators

Also, in this universal class, $\{0, \mathbb{Z}_2^0\}$ is nil and

$$\mathbf{USSS}(\{0, \mathbb{Z}_2^0\}) = \mathbf{SUUU}(\{0, \mathbb{Z}_2^0\}) = \{0, \mathbb{Z}_2^0\}.$$

[9] P. N. Stewart, 'Semi-simple radical classes' *Pacific J. Math.* **32** (1970), 249–254.

[10] R. Wiegandt, 'Radicals of rings defined by means of ring elements', *Sitzungsber. Österr. Akad. d. Wiss., Mathem.-naturw. Klasse*, **184** (1975), 117-125.

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This is a smaller radical-semisimple class than the universal class itself which is expected in the universal class of all associative rings.[9]

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What is emerging are radical and semisimple classes which account for the properties of the radical ideal in terms of the substructure of the class elements rather than a property that the elements of the ideal might have.[10]

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Thank you!