

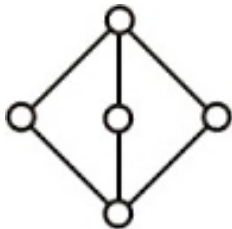
# On base radical and semisimple class operators for associative rings

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in collaboration with Dr Robert McDougall

University of the Sunshine Coast

November 24, 2017



I would like to acknowledge and thank the Australian Algebra Group, as well as the University of the Sunshine Coast, for providing funding to attend and present at the 1<sup>st</sup> Australian Algebra Conference, 2017.

# Background Preliminaries

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- ▶  $\mathcal{X}$  is *closed under extensions* if  $\mathcal{X}$  has the property that whenever  $I \triangleleft A \in \mathcal{A}$  and both  $I$  and  $A/I$  are elements of  $\mathcal{X}$ , then  $A$  is in  $\mathcal{X}$

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$\mathbf{U}(\mathcal{X}) = \{A \in \mathcal{A} \mid A \text{ has no nonzero homomorphic image in } \mathcal{X}\}$

$\mathbf{S}(\mathcal{X}) = \{A \in \mathcal{A} \mid A \text{ has no nonzero accessible subring in } \mathcal{X}\}$

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- ▶ For the trivial classes in  $\mathcal{A}$  we have  $\mathbf{U}(0) = \mathbf{S}(0) = \mathcal{A}$ , and  $\mathbf{U}(\mathcal{A}) = \mathbf{S}(\mathcal{A}) = 0$
- ▶ For all subclasses  $\mathcal{X} \subseteq \mathcal{A}$  we have  $\mathcal{X} \cap \mathbf{U}(\mathcal{X}) = \mathcal{X} \cap \mathbf{S}(\mathcal{X}) = 0$

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Meaningful outcomes generated by interchanging the operators  $\mathbf{U}$  and  $\mathbf{S}$  are called *dual* results.

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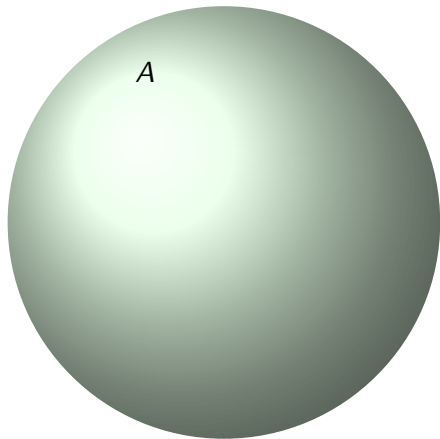
- (R1) If  $A \in \mathcal{R}$ , then every nonzero homomorphic image of  $A$  has a nonzero ideal in  $\mathcal{R}$ ;

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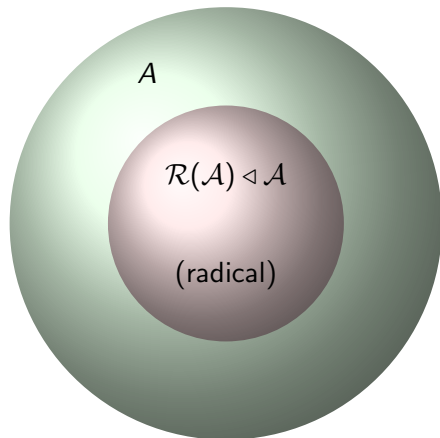
A class  $\mathcal{R}$  of rings is called a *radical class* whenever the following two axioms are satisfied:

- (R1) If  $A \in \mathcal{R}$ , then every nonzero homomorphic image of  $A$  has a nonzero ideal in  $\mathcal{R}$ ;
- (R2) If every nonzero homomorphic image of a ring  $A$  has a nonzero ideal in  $\mathcal{R}$ , then  $A \in \mathcal{R}$ .

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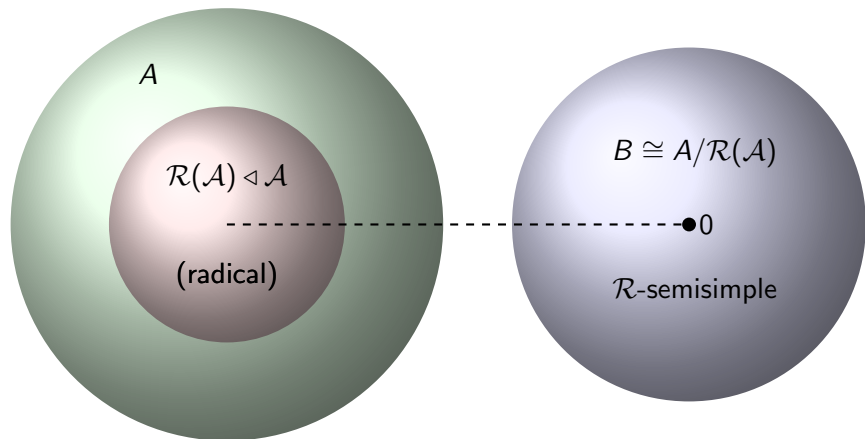


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The *base radical operator* **US** and *base semisimple operator* **SU**<sup>1</sup> are then

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<sup>1</sup>N. R. McConnell, R. G. McDougall, T. Stokes, 'On base radical and semisimple classes defined by class operators', *Acta Math. Hungar.* **138** (2013), 307–328

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The *base radical operator* **US** and *base semisimple operator* **SU**<sup>1</sup> are then

$\mathbf{US}(\mathcal{X}) = \{A \in \mathcal{A} \mid \text{every nonzero homomorphic image of } A \text{ has a nonzero accessible subring in } \mathcal{X}\}$

$\mathbf{SU}(\mathcal{X}) = \{A \in \mathcal{A} \mid \text{every nonzero accessible subring of } A \text{ has a nonzero homomorphic image in } \mathcal{X}\}$

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- ▶ A class  $\mathcal{X} \subseteq \mathcal{A}$  is a *base radical class* iff  $\mathcal{X} = \mathbf{US}(\mathcal{X})$ , and dually
- ▶  $\mathcal{X}$  is a *base semisimple class* iff  $\mathcal{X} = \mathbf{SU}(\mathcal{X})$ <sup>2</sup>

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For associative rings, Kurosh-Amitsur radical classes are precisely base radical classes and, since Kurosh-Amitsur semisimple classes are hereditary, their corresponding semisimple classes coincide.

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For associative rings, Kurosh-Amitsur radical classes are precisely base radical classes and, since Kurosh-Amitsur semisimple classes are hereditary, their corresponding semisimple classes coincide.

- ▶ A class  $\mathcal{X}$  is called *radical-semisimple* if  $\mathcal{X}$  is both a radical class and a semisimple class

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# Background Preliminaries

The operator semigroup  $RT_{\mathcal{A}}$

- ▶ contains all distinct operators composed of **U** and **S** that can act on subclasses of a universal class  $\mathcal{A}$

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Previously described for the universal class of two and four element cyclic groups and for the universal class of all simple associative rings. <sup>3</sup>

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- ▶ For all  $\mathbf{P}, \mathbf{Q} \in RT_{\mathcal{A}}$ ,  $\mathbf{P}=\mathbf{Q}$  iff  $\mathbf{P}(\mathcal{X}) = \mathbf{Q}(\mathcal{X})$  for all  $\mathcal{X} \subseteq \mathcal{A}$

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- ▶ For example, the universal class generated by associative ring  $\mathbb{Z}_4$  ( $\mathcal{A}_1 = \{0, \mathbb{Z}_4, \mathbb{Z}_2, \mathbb{Z}_2^0\}$ ) has associated operator semigroup

$RT_{\mathcal{A}_1} = \{\mathbf{U}, \mathbf{S}, \mathbf{UU}, \mathbf{US}, \mathbf{SU}, \mathbf{SS}, \mathbf{UUU}, \mathbf{USS}, \mathbf{USU}, \mathbf{SUS}, \mathbf{SSS}, \mathbf{USSS}\}$  with  $|RT_{\mathcal{A}_1}| = 12$

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- ▶  $\neg(\mathcal{X}) = \{A \in \mathcal{A} \mid A \notin \mathcal{X}\}$
- ▶ Denote the operator semigroup formed by operators **U**, **S** and  $\neg$  by  $RT_{\mathcal{A}}^*$

# On Base Radical and Semisimple Operators

## Proposition

If a subclass  $\mathcal{X} \subseteq \mathcal{A}$  is closed under extensions, then  $\mathbf{SU}(\mathcal{X}) = \mathbf{S}\neg(\mathcal{X})$ .

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## Proof.

Let a subclass  $\mathcal{X}$  of  $\mathcal{A}$  be closed under extensions with  $A \in \mathbf{SU}(\mathcal{X})$  such that  $A \notin \mathcal{X}$ , and  $C_1$  a nonzero accessible subring of  $A$ . Suppose  $C_1 \notin \mathcal{X}$ .  $C_1 \in \mathbf{SU}(\mathcal{X})$  since  $\mathbf{SU}(\mathcal{X})$  is hereditary, hence  $C_1$  has a nonzero homomorphic image  $D_1 \in \mathcal{X}$  such that  $D_1 \cong C_1/C_2$  for some nonzero ideal  $C_2$  of  $C_1$ .

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# On Base Radical and Semisimple Operators

Necessary and sufficient conditions for a class of associative rings to be semisimple are that the class be hereditary, closed under subdirect sums and closed under extensions.<sup>4</sup>

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## Proposition

*For all subclasses  $\mathcal{X} \subseteq \mathcal{A}$ ,  $\mathcal{X}$  is hereditary and closed under extensions iff  $\mathcal{X}$  is a semisimple class.*

## Proof.

If  $\mathcal{X}$  is hereditary and closed under extensions then

**SU**( $\mathcal{X}$ ) = **S** $\neg$ ( $\mathcal{X}$ ) =  $\mathcal{X}$  and so  $\mathcal{X}$  is semisimple. The converse is clear. □

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# On Base Radical and Semisimple Operators

## Theorem

For all subclasses  $\mathcal{X} \subseteq \mathcal{A}$ ,  $\mathbf{SUU}(\mathcal{X})$  is a radical-semisimple class.

## Proof.

Since  $\mathbf{UUU}(\mathcal{X})$  is a radical class for all  $\mathcal{X} \subseteq \mathcal{A}$  and hence closed under extensions. The first proposition gives

$\mathbf{SU}(\mathbf{UUU}(\mathcal{X})) = \mathbf{S}\neg(\mathbf{UUU}(\mathcal{X}))$  and so  $\mathbf{SUU}(\mathcal{X}) = \mathbf{S}\neg\mathbf{UUU}(\mathcal{X})$ .

The class  $\mathbf{S}\neg\mathbf{UUU}(\mathcal{X})$  is homomorphically closed and since any homomorphically closed semisimple class is a radical class <sup>5</sup>,

$\mathbf{SUU}(\mathcal{X})$  is a radical-semisimple class. □

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$\mathbf{SUU}(\mathcal{X})$  is a radical-semisimple class. □

## Proposition

For all subclasses  $\mathcal{X} \subseteq \mathcal{A}$ ,  $\mathbf{USS}(\mathcal{X}) = \mathbf{SUU}(\mathcal{X})$ .

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# On Base Radical and Semisimple Operators

Consider two properties a class  $\mathcal{X} \subseteq \mathcal{A}$  may have:

- (1\*) If  $A \in \mathcal{X}$  then  $A$  has a simple nonzero homomorphic image in  $\mathcal{X}$ ,
- (2\*) If  $A \in \mathcal{X}$  then  $A$  has a simple nonzero accessible subring in  $\mathcal{X}$ .

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## Proposition

*For all subclasses  $\mathcal{X} \subseteq \mathcal{A}$ , the following are equivalent:*

- (i)  $\mathcal{X} = \mathbf{US}(\mathcal{X}) = \mathbf{SU}(\mathcal{X})$ ,
- (ii)  $\mathcal{X} = \mathbf{USSS}(\mathcal{X}) = \mathbf{SUUU}(\mathcal{X})$ ,
- (iii)  $\mathcal{X}$  is a radical class with condition (2\*),
- (iv)  $\mathcal{X}$  is a semisimple class with condition (1\*).

# On Base Radical and Semisimple Operators

## Proposition

A radical class  $\mathcal{R}$  is hereditary iff  $\mathcal{R}$  is of the form **USS**( $\mathcal{X}$ ) for some subclass  $\mathcal{X}$  of  $\mathcal{A}$ .

## Proof.

If a radical class  $\mathcal{R}$  is hereditary then  $\mathcal{R}$  has condition (2\*) and, by the previous proposition,  $\mathcal{R} = \mathbf{USSS}(\mathcal{R})$  and so  $\mathcal{R}$  is of the form **USS**( $\mathcal{X}$ ). The converse is clear from the previous proposition.  $\square$



# On Base Radical and Semisimple Operators

For the universal class of all associative rings the 'concrete radical classes', including the Jacobson radical, are all hereditary<sup>6</sup> and it follows that these are all radical-semisimple classes in this setting.

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## Proposition

*If condition (2\*) holds for a subclass  $\mathcal{X} \subseteq \mathcal{A}$  then the radical class  $\mathbf{US}(\mathcal{X})$  is a radical-semisimple class. Dually, if condition (1\*) holds for a subclass  $\mathcal{X} \subseteq \mathcal{A}$  then the semisimple class  $\mathbf{SU}(\mathcal{X})$  is a radical-semisimple class.*

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## On Base Radical and Semisimple Operators

It is known that the largest hereditary subclass of a radical class is a radical class<sup>7</sup>, and in this finite setting we can extend this to

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## Theorem

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For all subclasses  $\mathcal{X} \subseteq \mathcal{A}$ ,  $\mathbf{SUUU}(\mathcal{X}) = \mathbf{S}\neg\mathbf{USSS}(\mathcal{X}) = \mathbf{S}\neg\mathbf{USU}\neg(\mathcal{X}) = \mathbf{S}\neg\mathbf{US}(\mathcal{X}) = \mathbf{S}\neg\mathbf{UUU}(\mathcal{X}) = \mathbf{U}\neg\mathbf{SUUU}(\mathcal{X}) = \mathbf{U}\neg\mathbf{SUS}\neg(\mathcal{X}) = \mathbf{U}\neg\mathbf{SU}(\mathcal{X}) = \mathbf{USSS}(\mathcal{X})$ .

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## On Base Radical and Semisimple Operators

For any subclass  $\mathcal{X} \subseteq \mathcal{A}$ ,  $\mathbf{S}(\mathcal{X})$  is a semisimple class when  $\mathcal{X}$  is homomorphically closed<sup>8</sup> or hereditary.

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## Proposition

For a subclass  $\mathcal{X} \subseteq \mathcal{A}$ , if  $\mathcal{X} \subseteq \mathbf{SU}(\mathcal{X})$  then  $\mathbf{S}(\mathcal{X})$  is a semisimple class.

## Proof.

If a subclass  $\mathcal{X}$  of  $\mathcal{A}$  is contained in  $\mathbf{SU}(\mathcal{X})$ , then  $\mathbf{SSU}(\mathcal{X}) \subseteq \mathbf{S}(\mathcal{X}) \subseteq \mathbf{SSS}(\mathcal{X})$ . Since  $\mathbf{SSU}(\mathcal{X}) = \mathbf{SSS}(\mathcal{X})$  it follows that  $\mathbf{S}(\mathcal{X}) = \mathbf{SSS}(\mathcal{X})$ , showing  $\mathbf{S}(\mathcal{X})$  is a semisimple class.  $\square$

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## On Base Radical and Semisimple Operators

The correspondence theorem shows that, for all  $A \in \mathcal{A}$ , every accessible subring of a homomorphic image of  $A$  is a homomorphic image of an accessible subring of  $A$ . We can dualise this result to an extent for finite associative rings as follows.



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### Proposition

*For all  $A \in \mathcal{A}$ , every simple homomorphic image of an accessible subring of  $A$  is a simple accessible subring of a homomorphic image of  $A$ .*

### Proof.

Note that since  $\mathbf{USSS}(\mathcal{X}) = \mathbf{SUUU}(\mathcal{X})$  for all  $\mathcal{X} \subseteq \mathcal{A}$  we have  $\mathbf{USSS}(\mathcal{X}) = \mathbf{SUUU}(\mathcal{X}) = \mathbf{S}\neg\mathbf{UU}(\mathcal{X}) \subseteq \mathbf{U}\neg\mathbf{SU}(\mathcal{X}) \subseteq \mathbf{U}\neg\mathbf{SS}(\mathcal{X}) \subseteq \mathbf{USSS}(\mathcal{X})$  and so all classes are equal. Suppose  $C$  is a simple homomorphic image of an accessible subring of  $A \in \mathcal{A}$  such that  $C$  is not the simple accessible subring of a homomorphic image of  $A$ . Then  $A \in \neg\mathbf{S}\neg\mathbf{UUU}(\{C\})$  but  $A \notin \neg\mathbf{U}\neg\mathbf{SSS}(\{C\})$ . Hence  $\mathbf{S}\neg\mathbf{UU}(\mathcal{X}) \neq \mathbf{U}\neg\mathbf{SS}(\mathcal{X})$ , a contradiction. □

# On Base Radical and Semisimple Operators

For any subclass  $\mathcal{X} \subseteq \mathcal{A}$ ,

$$\mathcal{X}_s = \{A \in \mathcal{A} \mid A \in \mathcal{X} \text{ and } A \text{ is simple}\}$$

$\mathcal{X}_s$  is hereditary and homomorphically closed and generates coinciding lower radical class and semisimple closure.

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## Proposition

For all subclasses  $\mathcal{X} \subseteq \mathcal{A}$ ,  $\mathbf{US}(\mathcal{X}_s) = \mathbf{SU}(\mathcal{X}_s)$ .

## On Base Radical and Semisimple Operators

We now determine the maximum order of  $RT_{\mathcal{A}}$  and  $RT_{\mathcal{A}}^*$  for universal classes of finite associate rings, including a complete listing of possible radical and semisimple class operators.

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## Theorem

*For the operator semigroup  $RT_{\mathcal{A}}$ ,  $|RT_{\mathcal{A}}| \leq 12$ .*

## Proof.

Starting with the generating classes  $\mathbf{U}(\mathcal{X})$  and  $\mathbf{S}(\mathcal{X})$  and applying  $\mathbf{U}$  and  $\mathbf{S}$  to each in turn we have  $\mathbf{UU}(\mathcal{X})$ ,  $\mathbf{US}(\mathcal{X})$ ,  $\mathbf{SU}(\mathcal{X})$  and  $\mathbf{SS}(\mathcal{X})$ .

Repeating generates classes  $\mathbf{UUU}(\mathcal{X})$ ,  $\mathbf{SUU}(\mathcal{X})$ ,  $\mathbf{USU}(\mathcal{X})$ ,  $\mathbf{SUS}(\mathcal{X})$  and  $\mathbf{SSS}(\mathcal{X})$ , since other possibilities collapse. A further application of  $\mathbf{S}$  gives the last distinct class  $\mathbf{SUUU}(\mathcal{X})$ . In summary

$RT_{\mathcal{A}} = \{\mathbf{U}, \mathbf{S}, \mathbf{UU}, \mathbf{US}, \mathbf{SU}, \mathbf{SS}, \mathbf{UUU}, \mathbf{SUU}, \mathbf{USU}, \mathbf{SUS}, \mathbf{SSS}, \mathbf{SUUU}\}$

showing that the order of  $RT_{\mathcal{A}}$  is at most 12.  $\square$

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showing that the order of  $RT_{\mathcal{A}}$  is at most 12.  $\square$

This is the semigroup mentioned earlier for the universal class  $\mathcal{A}_1$ .

## On Base Radical and Semisimple Operators

Using a similar proof strategy we can determine an upper bound for the order of the semigroup generated by  $\mathbf{U}$ ,  $\mathbf{S}$  and  $\neg$ .

# On Base Radical and Semisimple Operators

Using a similar proof strategy we can determine an upper bound for the order of the semigroup generated by **U**, **S** and  $\neg$ .

## Theorem

For the semigroup  $RT_{\mathcal{A}}^*$ ,  $|RT_{\mathcal{A}}^*| \leq 46$ .

The complete list  $RT_{\mathcal{A}}^* = \{\mathbf{U}, \mathbf{S}, \neg, \mathbf{UU}, \mathbf{SU}, \neg\mathbf{U}, \mathbf{US}, \mathbf{SS}, \neg\mathbf{S}, \mathbf{U}\neg, \mathbf{S}\neg, \neg\neg, \mathbf{UUU}, \mathbf{SUU}, \neg\mathbf{UU}, \mathbf{USU}, \mathbf{SSS}, \neg\mathbf{SU}, \mathbf{S}\neg\mathbf{U}, \mathbf{SUS}, \neg\mathbf{US}, \neg\mathbf{SS}, \mathbf{U}\neg\mathbf{S}, \mathbf{SU}\neg, \neg\mathbf{U}\neg, \mathbf{US}\neg, \neg\mathbf{S}\neg, \mathbf{SUUU}, \neg\mathbf{UUU}, \neg\mathbf{SUU}, \neg\mathbf{USU}, \neg\mathbf{SSS}, \neg\mathbf{S}\neg\mathbf{U}, \neg\mathbf{SUS}, \neg\mathbf{U}\neg\mathbf{S}, \mathbf{USU}\neg, \neg\mathbf{SU}\neg, \mathbf{S}\neg\mathbf{U}\neg, \mathbf{SUS}\neg, \neg\mathbf{US}\neg, \mathbf{U}\neg\mathbf{S}\neg, \neg\mathbf{SUUU}, \neg\mathbf{USU}\neg, \neg\mathbf{S}\neg\mathbf{U}\neg, \neg\mathbf{SUS}\neg, \neg\mathbf{U}\neg\mathbf{S}\neg\}$ .



## On Base Radical and Semisimple Operators

Considering  $\mathcal{A}_1 = \{0, \mathbb{Z}_4, \mathbb{Z}_2, \mathbb{Z}_2^0\}$ , for all  $A \in \mathcal{A}_1$ , every homomorphic image of each accessible subring of  $A$  is an accessible subring of a homomorphic image of  $A$ , and so for all subclasses  $\mathcal{X} \subseteq \mathcal{A}_1$  we have  $\mathbf{S}\neg\mathbf{U}(\mathcal{X}) = \mathbf{U}\neg\mathbf{S}(\mathcal{X})$ .

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Other equalities based on properties of the elements of  $\mathcal{A}_1$  hold, and we have

$$RT_{\mathcal{A}_1}^* = \{\mathbf{U}, \mathbf{S}, \neg, \mathbf{UU}, \mathbf{SU}, \neg\mathbf{U}, \mathbf{US}, \mathbf{SS}, \neg\mathbf{S}, \mathbf{U}\neg, \mathbf{S}\neg, \neg\neg, \mathbf{UUU}, \mathbf{SUU}, \mathbf{USU}, \mathbf{SSS}, \neg\mathbf{SU}, \mathbf{S}\neg\mathbf{U}, \mathbf{SUS}, \neg\mathbf{US}, \neg\mathbf{U}\neg, \neg\mathbf{S}\neg, \mathbf{SUUU}, \neg\mathbf{SUU}, \neg\mathbf{USU}, \neg\mathbf{S}\neg\mathbf{U}, \neg\mathbf{SUS}, \mathbf{S}\neg\mathbf{U}\neg, \neg\mathbf{SUUU}, \neg\mathbf{S}\neg\mathbf{U}\neg\}$$

with  $|RT_{\mathcal{A}_1}^*| = 30$ .

## On Base Radical and Semisimple Operators

Also, in this universal class,  $\{0, \mathbb{Z}_2^0\}$  is nil and

$$\mathbf{USSS}(\{0, \mathbb{Z}_2^0\}) = \mathbf{SUUU}(\{0, \mathbb{Z}_2^0\}) = \{0, \mathbb{Z}_2^0\}.$$

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<sup>9</sup>P. N. Stewart, 'Semi-simple radical classes' *Pacific J. Math.* **32** (1970), 249–254.

<sup>10</sup>R. Wiegandt, 'Radicals of rings defined by means of ring elements', *Sitzungsber. Österr. Akad. d. Wiss., Mathem.-naturw. Klasse*, **184** (1975), 117-125.

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This is a smaller radical-semisimple class than the universal class itself which is expected in the universal class of all associative rings.<sup>9</sup>

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What is emerging are radical and semisimple classes which account for the properties of the radical ideal in terms of the substructure of the class elements rather than a property that the elements of the ideal might have.<sup>10</sup>

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Thank you!

