The lattice of subgraphs

Christopher J. Taylor
La Trobe University

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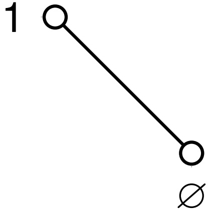

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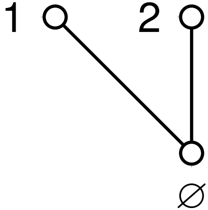

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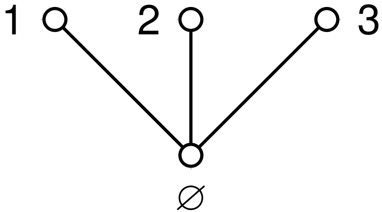

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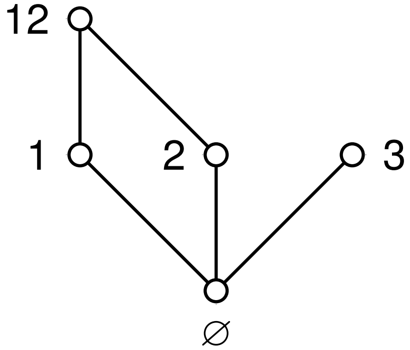

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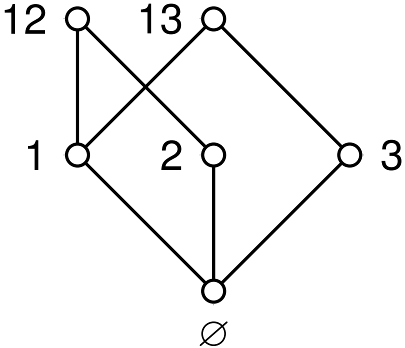

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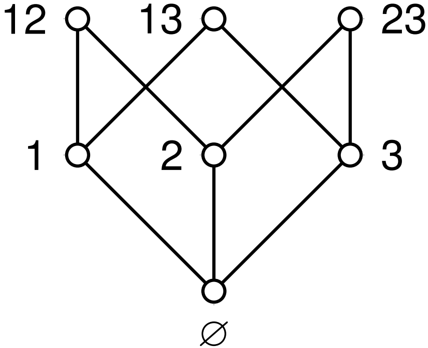

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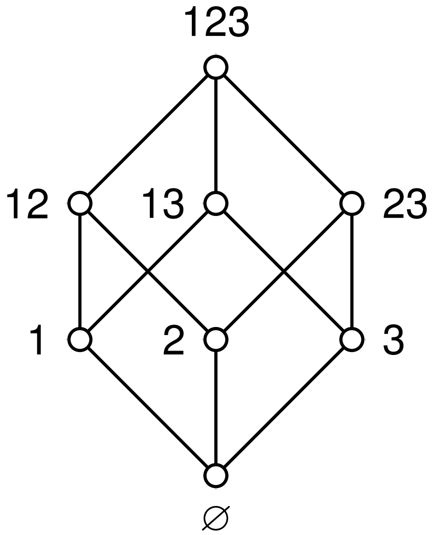

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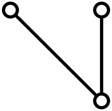

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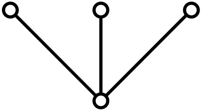

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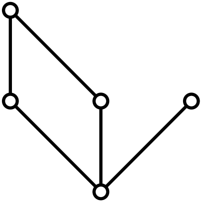

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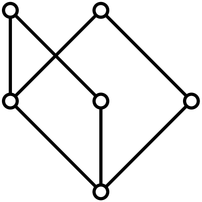

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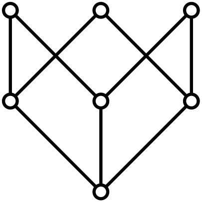

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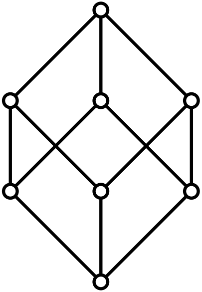

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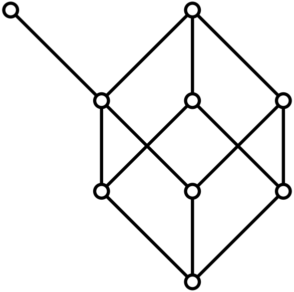

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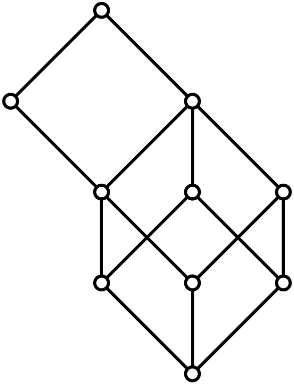

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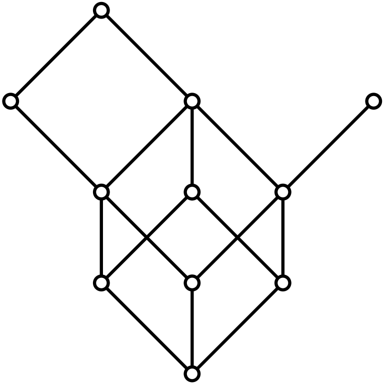

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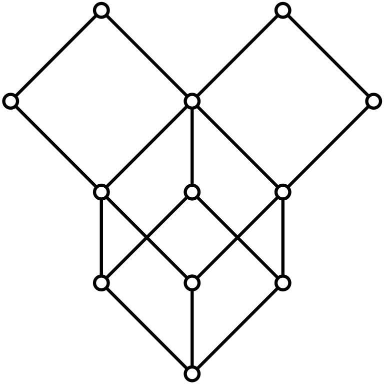

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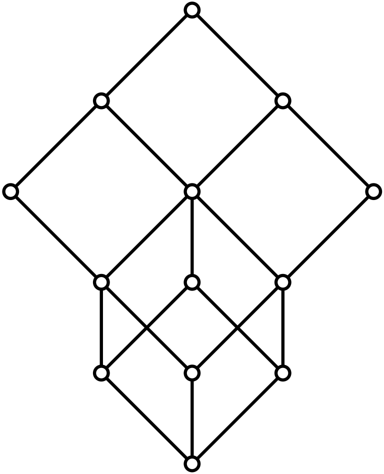
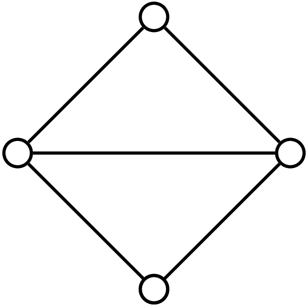


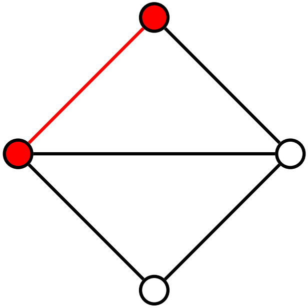
Subgraph lattices

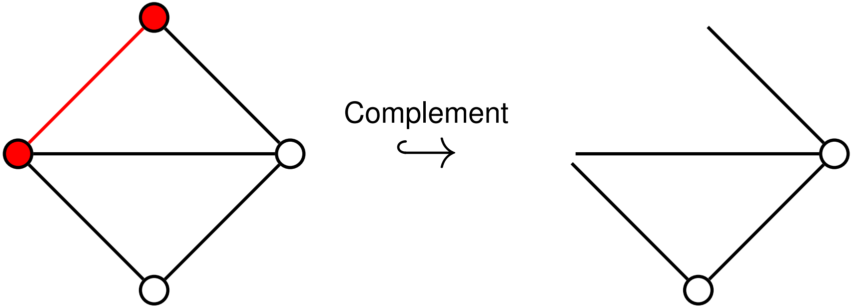
Let $G = \langle V, E \rangle$ be a graph. The set of all subgraphs of G, ordered by inclusion, is a bounded distributive lattice, where

$$\langle V_1, E_1 \rangle \vee \langle V_2, E_2 \rangle = \langle V_1 \cup V_2, E_1 \cup E_2 \rangle$$
$$\langle V_1, E_1 \rangle \wedge \langle V_2, E_2 \rangle = \langle V_1 \cap V_2, E_1 \cap E_2 \rangle.$$

The bounds are given by $0 = \langle \varnothing, \varnothing \rangle$ and $1 = G$. The lattice will be denoted by $\mathcal{S}(G)$.



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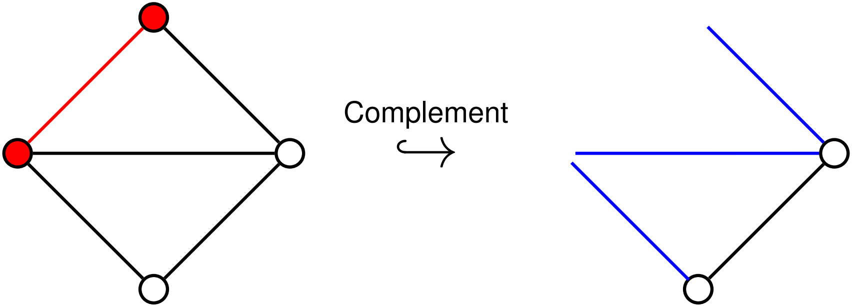
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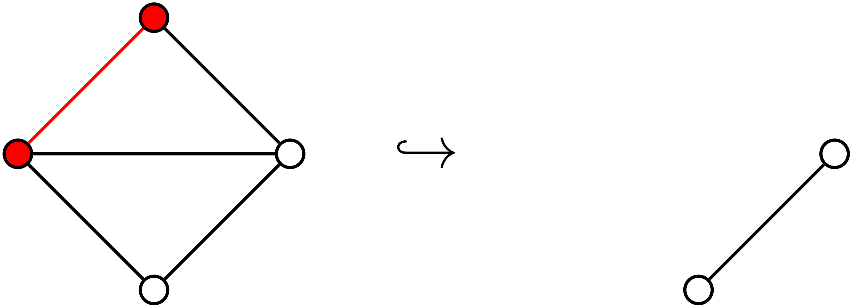

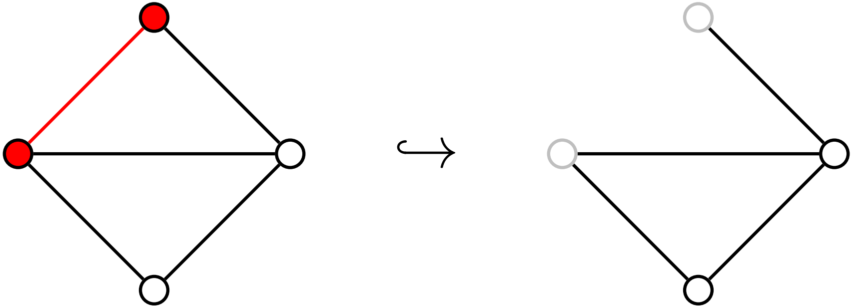

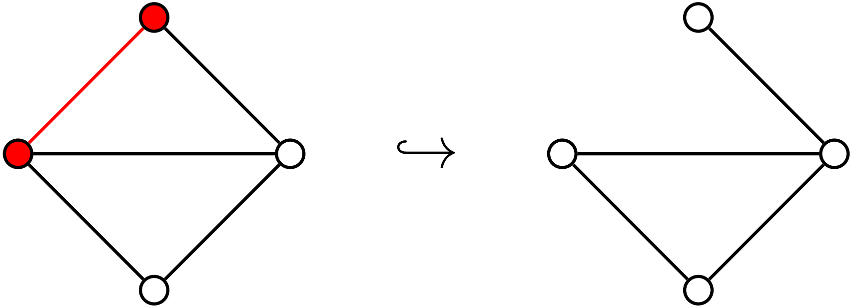

Pseudocomplements

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The lattice of open sets of a topological space X is pseudocomplemented: if U is an open set, then $\neg U = \mathrm{int}(X \backslash U)$.

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Similarly, a unary operation $\sim$ is a *dual pseudocomplement* if

$$\sim x = \min\{z \in L \mid x \vee z = 1\}.$$

The algebra of subgraphs

Definition

A (distributive) double p-algebra is an algebra $\mathbf{A} = \langle A, \vee, \wedge, \neg, \sim, 0, 1 \rangle$ such that

1. $\langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded (distributive) lattice,
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Theorem

Let G be a graph. The lattice $\mathcal{S}(G)$ underlies a distributive double p-algebra.

Pseudocomplements of subgraphs

Take the set complement of the subgraph and abandon the extra edges. Formally, for a graph $G = \langle V, E \rangle$ and a subgraph $H = \langle V', E' \rangle$,

$$\neg H = \langle V \setminus V', \{e \in E \setminus E' \mid (\forall x \in e)\ x \in V \setminus V'\} \rangle.$$

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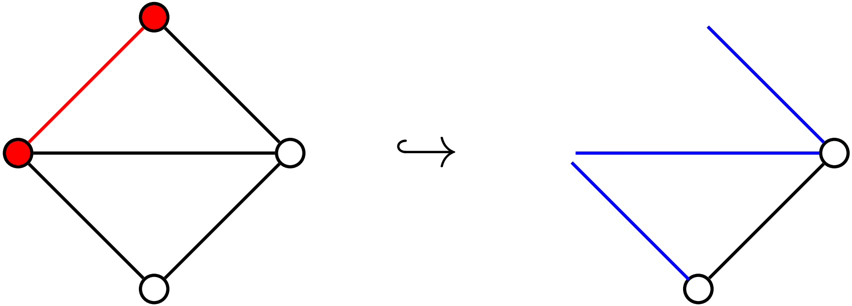

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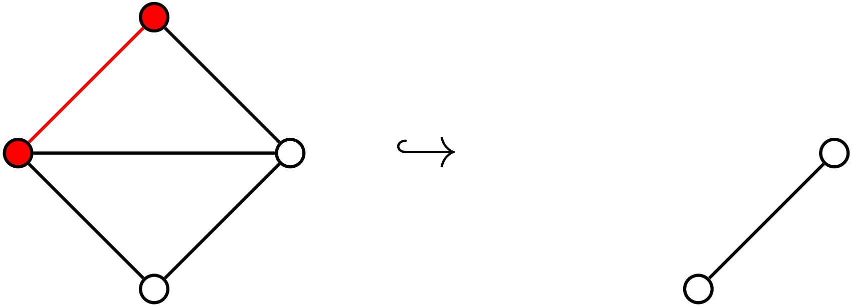

Dual pseudocomplements of subgraphs

Just add the missing vertices back. Formally, for a graph $G = \langle V, E \rangle$ and a subgraph $H = \langle V', E' \rangle$,

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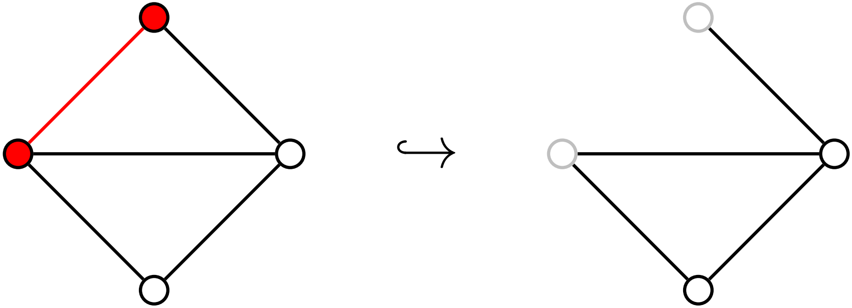

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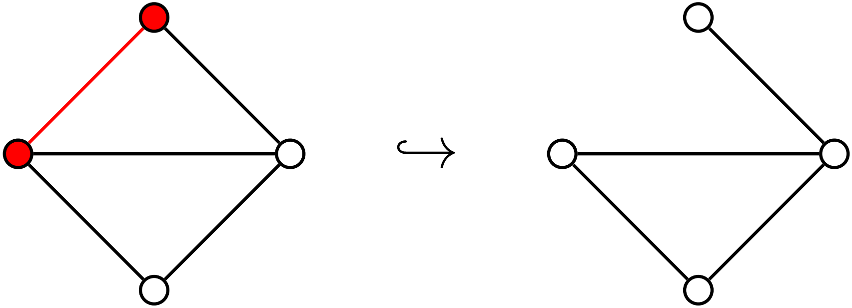

Regular double p-algebras

An algebra **A** is (*congruence-*)*regular* if, whenever two congruences on **A** share a block, they are actually the same congruence.

- e.g., groups

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Theorem (Varlet, 1972; Katriňák, 1973)

Let **A** *be a double p-algebra. The following are equivalent*:

1. **A** *is regular*;
2. *every prime ideal of* **A** *is minimal or maximal*;
3. *for all* $a, b \in A$, *if* $\neg a = \neg b$ *and* $\sim a = \sim b$, *then* $a = b$;
4. **A** *is distributive and* $\sim a \wedge a \leq \neg b \wedge b$, *for all* $a, b \in A$.

The representation theorem — finite version

Theorem (T., 2016)

Let G be a graph. Then $\mathcal{S}(G)$ is a regular double p-algebra.

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Theorem (T., 2016)

Let $\mathbf{L}$ *be a finite lattice. The following are equivalent.*

1. $\mathbf{L}$ *is the underlying lattice of a regular double p-algebra.*
2. $\mathbf{L} \cong \mathcal{S}(G)$ *for some finite incidence structure G.*
3. $\mathbf{L} \cong \mathcal{P}(B) \times \mathcal{S}(G)$ *for some finite set B and incidence structure G.*

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It is widely conjectured that the answer is *yes*, provided that the graphs have more than two vertices.

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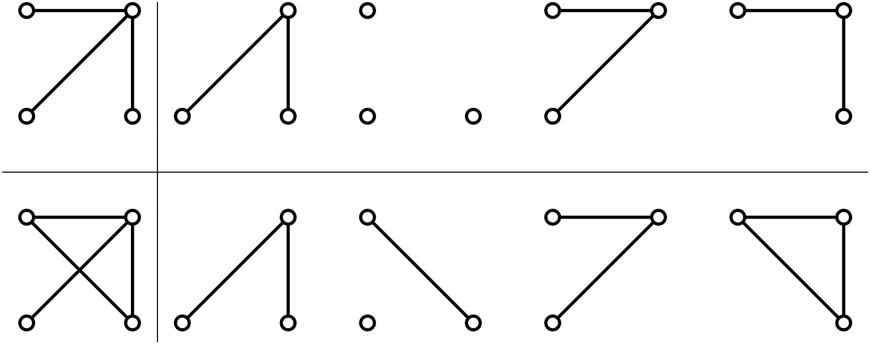

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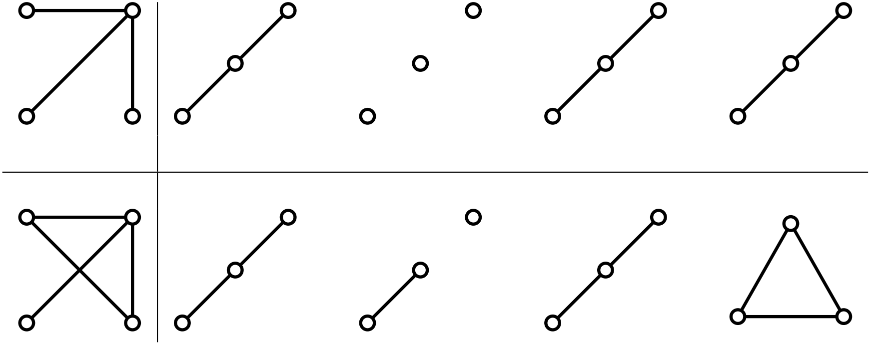

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Let G be a graph.

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- Let $H = \neg\langle \{v\}, \varnothing \rangle$.
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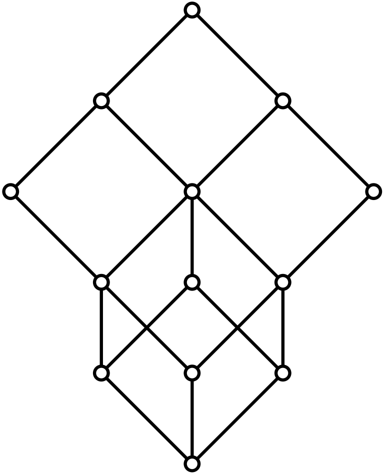
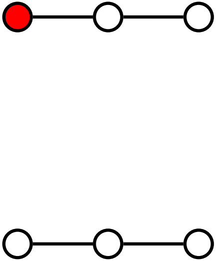
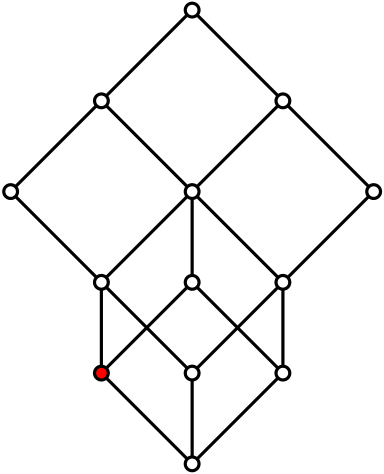
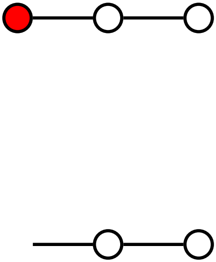
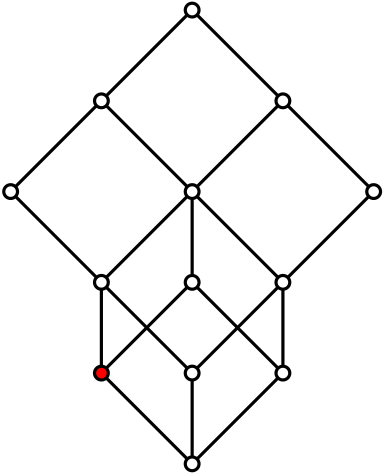
Hence, knowing the deck of G is the same as knowing the downsets of pseudocomplements of atoms in $\mathcal{S}(G)$.

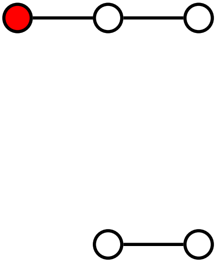


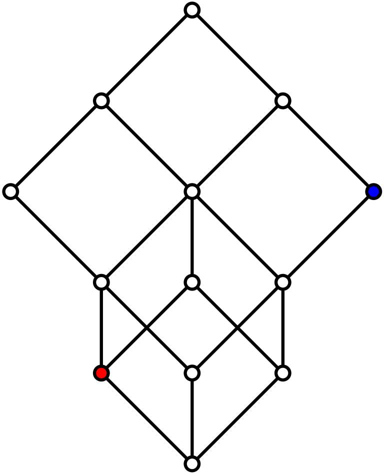


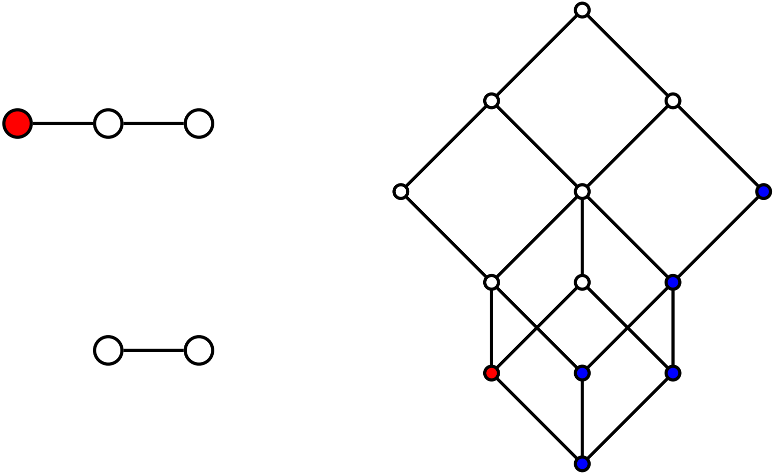












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1. I is a downset and closed under $\vee$,
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Proposition

Let G be a finite graph. For every vertex v in G, the set $\downarrow\neg\langle\{v\},\varnothing\rangle$ is a *minimal* prime ideal. Moreover, every minimal prime ideal is of this form.

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Definition

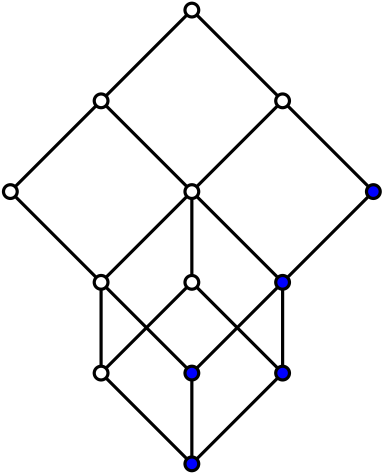
Let $\mathbf{L}$ be a lattice and let $I \subseteq L$. Then I is a *prime ideal* if

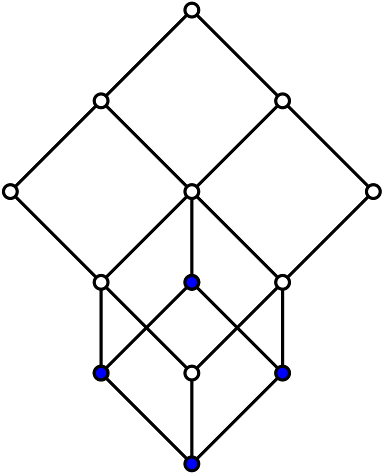
1. I is a downset and closed under $\vee$,
2. if $x \wedge y \in I$, then $x \in I$ or $y \in I$.

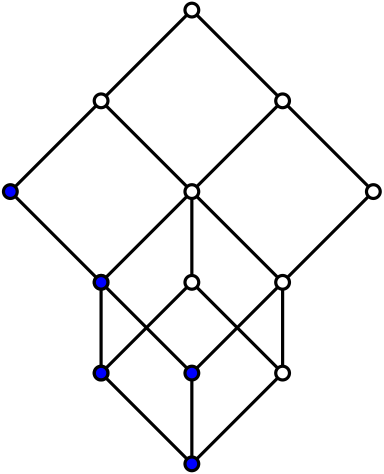
Proposition

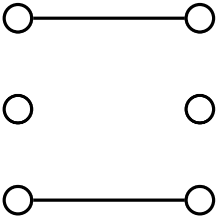
Let G be a finite graph. For every vertex v in G, the set $\downarrow\neg\langle\{v\}, \varnothing\rangle$ is a *minimal* prime ideal. Moreover, every minimal prime ideal is of this form.

- Let the deck of a lattice be the set of its minimal prime ideals and define reconstructions analogously.









Translated into the world of lattices

Lattice Reconstruction Problem

Let $\mathcal{K}$ be the class of all lattices **L** such that:

1. **L** is a finite distributive lattice.
2. Every prime ideal of **L** is minimal or maximal.
3. Every join-irreducible element of **L** is either an atom or has exactly two atoms below it.

If two lattices in $\mathcal{K}$ are reconstructions of each other, then are they isomorphic?

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