

The lattice of subgraphs

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La Trobe University

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University of Technology Sydney
November 2017

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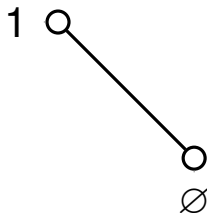
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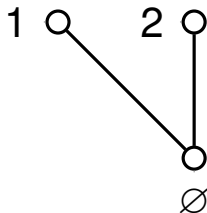
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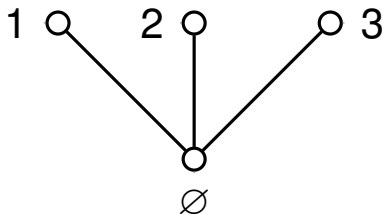
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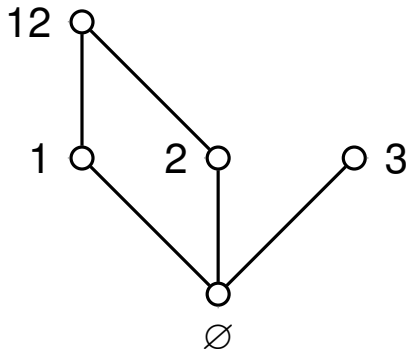
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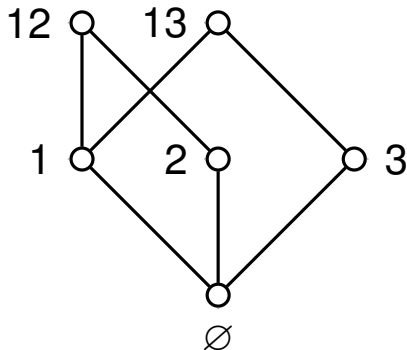
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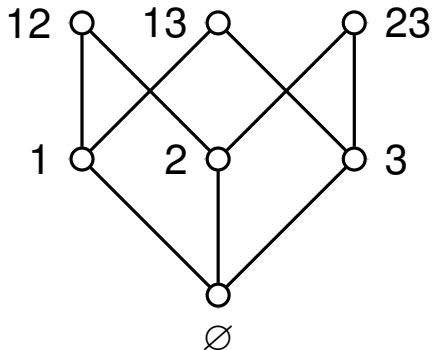
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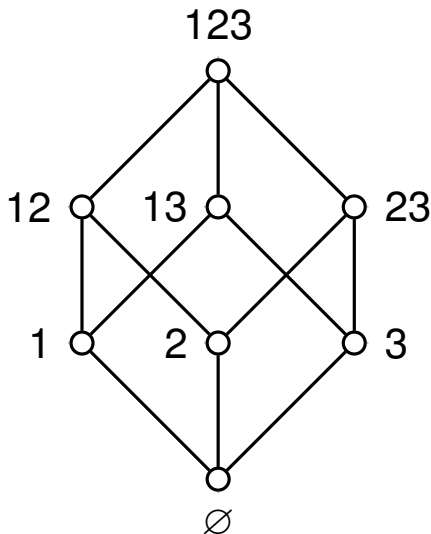
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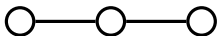
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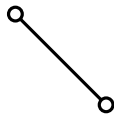
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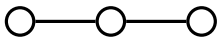
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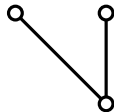
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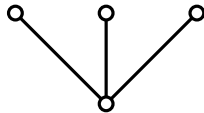
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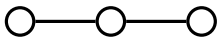
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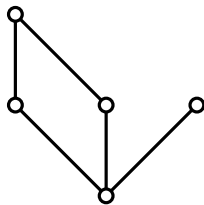
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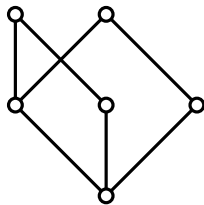
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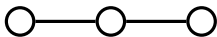
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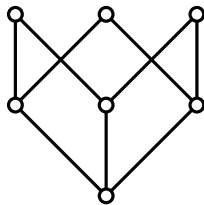
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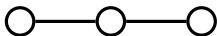
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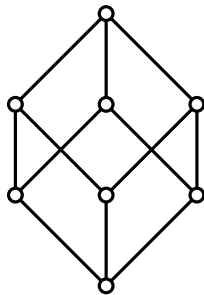
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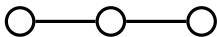
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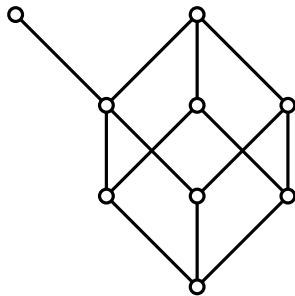
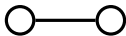
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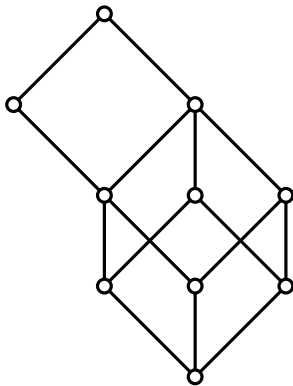
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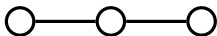
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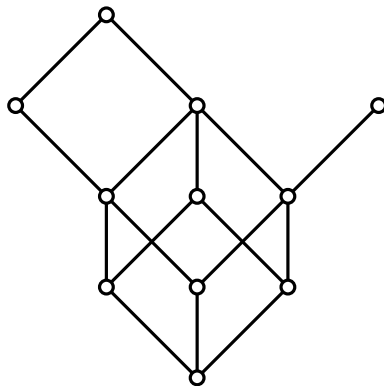
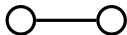
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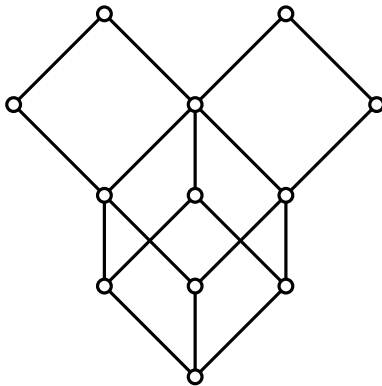
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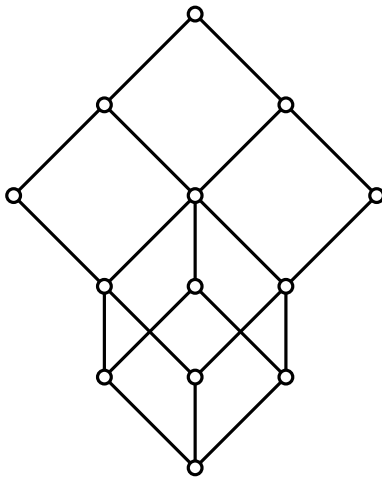
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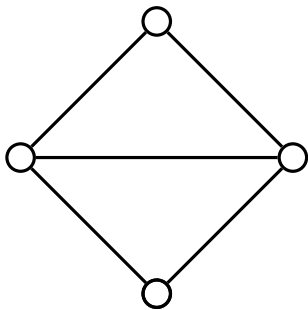


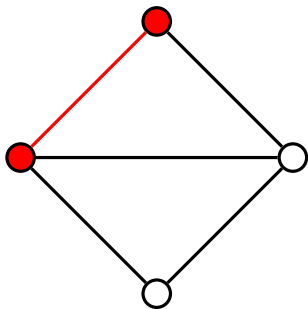
Subgraph lattices

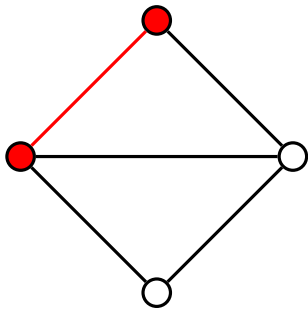
Let $G = \langle V, E \rangle$ be a graph. The set of all subgraphs of G , ordered by inclusion, is a bounded distributive lattice, where

$$\begin{aligned}\langle V_1, E_1 \rangle \vee \langle V_2, E_2 \rangle &= \langle V_1 \cup V_2, E_1 \cup E_2 \rangle \\ \langle V_1, E_1 \rangle \wedge \langle V_2, E_2 \rangle &= \langle V_1 \cap V_2, E_1 \cap E_2 \rangle.\end{aligned}$$

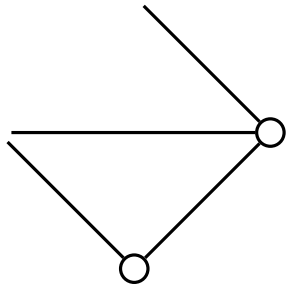
The bounds are given by $0 = \langle \emptyset, \emptyset \rangle$ and $1 = G$. The lattice will be denoted by $\mathcal{S}(G)$.

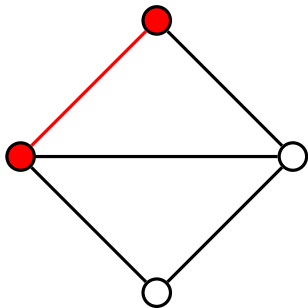




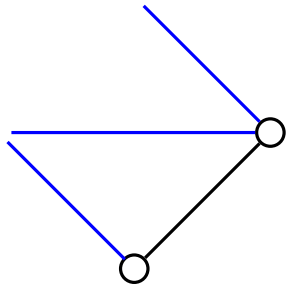


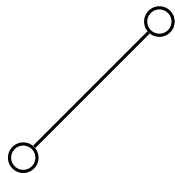
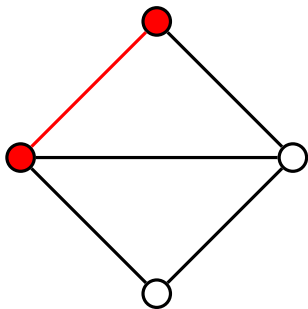
Complement
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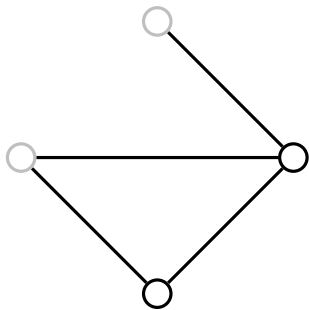
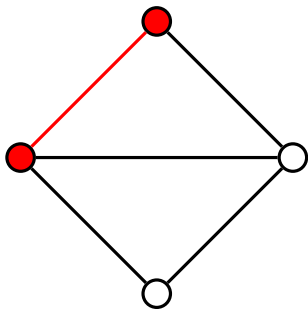


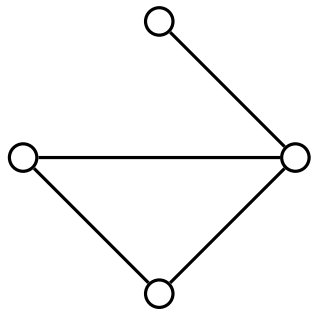
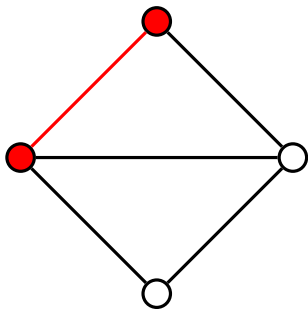


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The lattice of open sets of a topological space X is pseudocomplemented: if U is an open set, then $\neg U = \text{int}(X \setminus U)$.

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Similarly, a unary operation \sim is a *dual pseudocomplement* if

$$\sim x = \min\{z \in L \mid x \vee z = 1\}.$$

The algebra of subgraphs

Definition

A (distributive) double p-algebra is an algebra $\mathbf{A} = \langle \mathbf{A}, \vee, \wedge, \neg, \sim, 0, 1 \rangle$ such that

- 1 $\langle \mathbf{A}, \vee, \wedge, 0, 1 \rangle$ is a bounded (distributive) lattice,
- 2 \neg is the pseudocomplement,
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Theorem

Let G be a graph. The lattice $\mathcal{S}(G)$ underlies a distributive double p -algebra.

Pseudocomplements of subgraphs

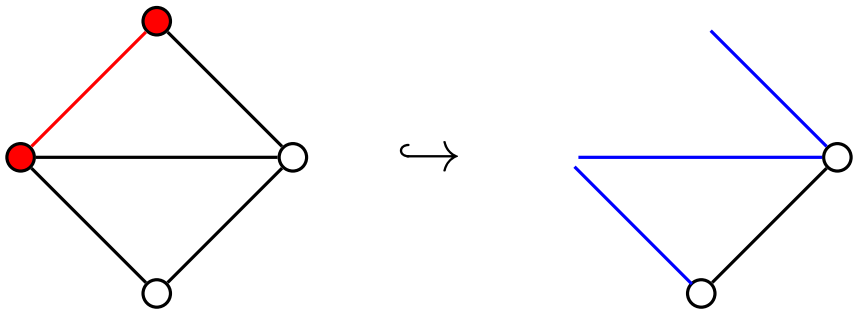
Take the set complement of the subgraph and abandon the extra edges. Formally, for a graph $G = \langle V, E \rangle$ and a subgraph $H = \langle V', E' \rangle$,

$$\neg H = \langle V \setminus V', \{e \in E \setminus E' \mid (\forall x \in e) x \in V \setminus V'\} \rangle.$$

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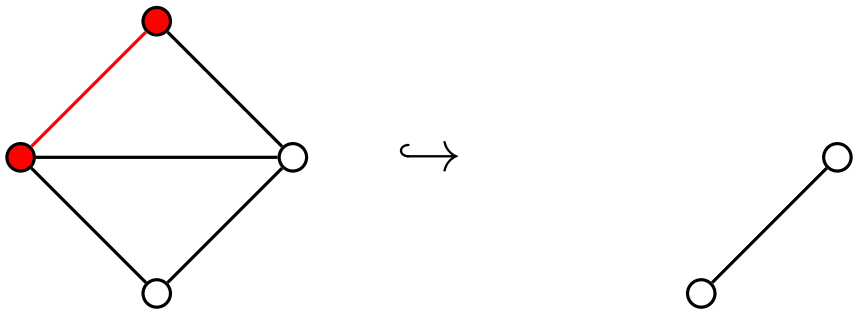
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Dual pseudocomplements of subgraphs

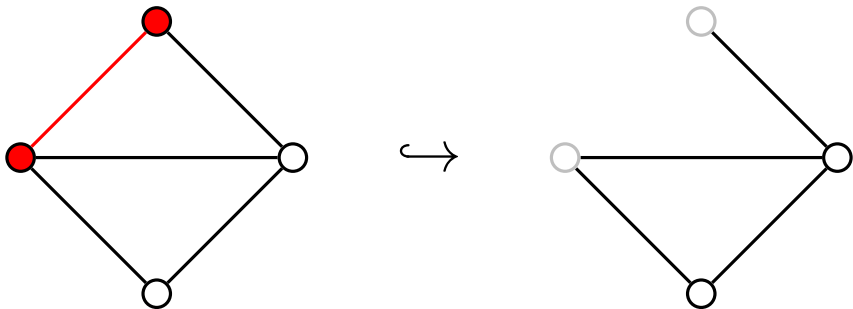
Just add the missing vertices back. Formally, for a graph $G = \langle V, E \rangle$ and a subgraph $H = \langle V', E' \rangle$,

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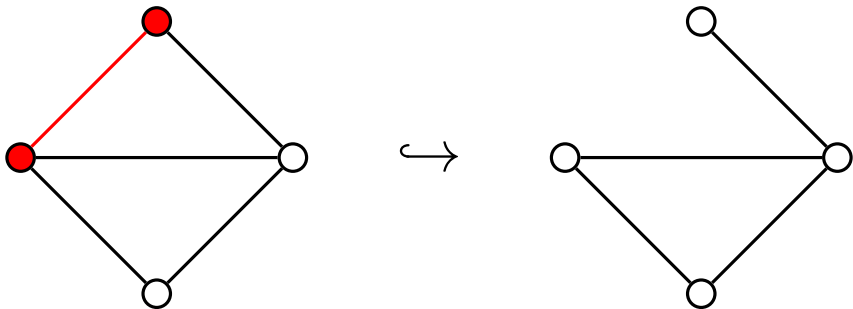
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Regular double p-algebras

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Theorem (Varlet, 1972; Katriňák, 1973)

Let \mathbf{A} be a double p-algebra. The following are equivalent:

- 1 \mathbf{A} is regular;
- 2 every prime ideal of \mathbf{A} is minimal or maximal;
- 3 for all $a, b \in A$, if $\neg a = \neg b$ and $\sim a = \sim b$, then $a = b$;
- 4 \mathbf{A} is distributive and $\sim a \wedge a \leq \neg b \wedge b$, for all $a, b \in A$.

The representation theorem — finite version

Theorem (T., 2016)

Let G be a graph. Then $S(G)$ is a regular double p -algebra.

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Theorem (T., 2016)

Let \mathbf{L} be a finite lattice. The following are equivalent.

- 1 \mathbf{L} is the underlying lattice of a regular double p -algebra.*
- 2 $\mathbf{L} \cong S(G)$ for some finite incidence structure G .*
- 3 $\mathbf{L} \cong \mathcal{P}(B) \times S(G)$ for some finite set B and incidence structure G .*

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If G is a graph and H is a reconstruction of G , is H isomorphic to G ?

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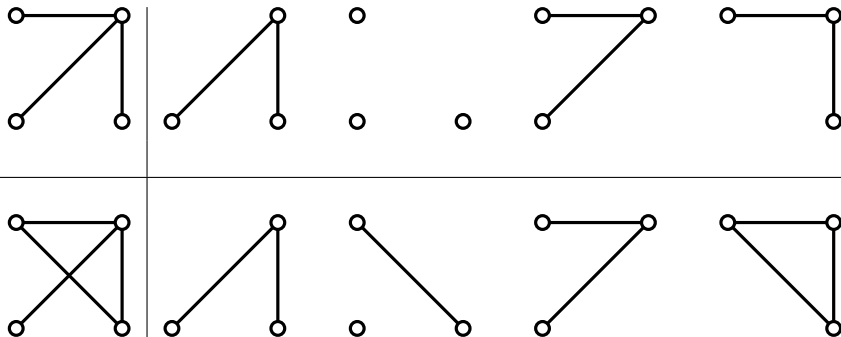
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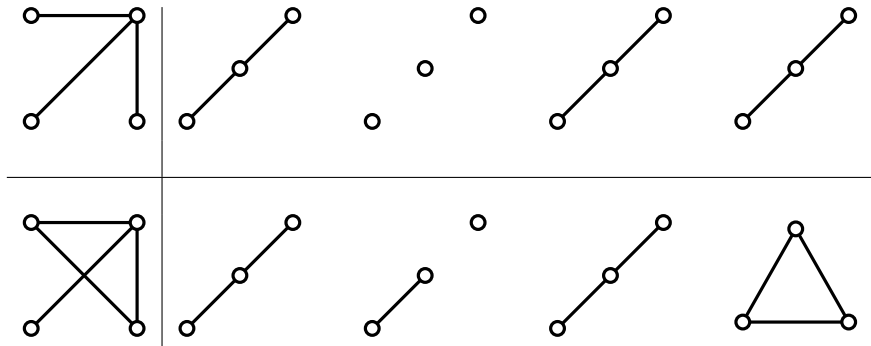
If G is a graph and H is a reconstruction of G , is H isomorphic to G ?

It is widely conjectured that the answer is yes, provided that the graphs have more than two vertices.

Example



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Translated into the world of lattices

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- The subgraph $\langle \{v\}, \emptyset \rangle$ is an atom: it covers the bottom of $\mathcal{S}(G)$.

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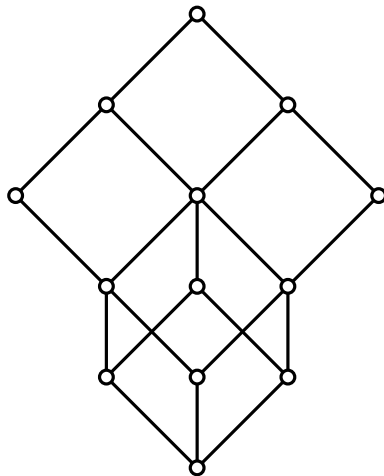
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- The structure of H is determined exactly by the structure of the downset $\downarrow H$.

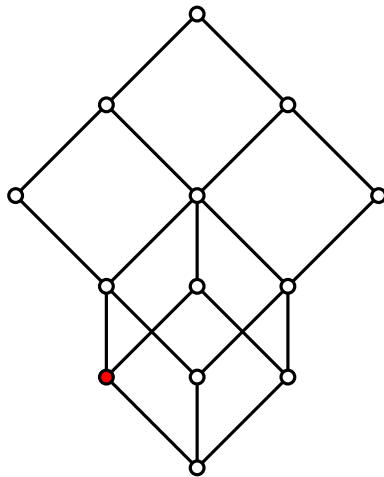
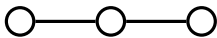
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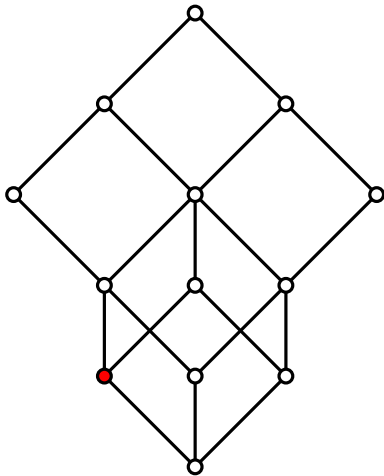
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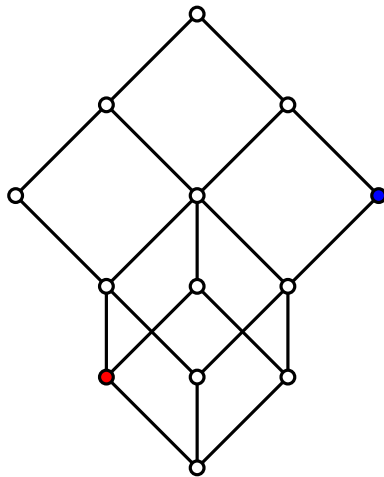
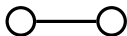
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- The structure of H is determined exactly by the structure of the downset $\downarrow H$.

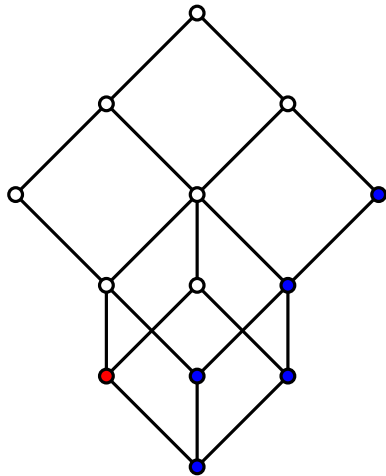
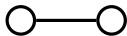
Hence, knowing the deck of G is the same as knowing the downsets of pseudocomplements of atoms in $\mathcal{S}(G)$.











Translated into the world of lattices

We can make this more precise.

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Definition

Let \mathbf{L} be a lattice and let $I \subseteq L$. Then I is a *prime ideal* if

- 1 I is a downset and closed under \vee ,
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Proposition

Let G be a finite graph. For every vertex v in G , the set $\downarrow \neg \langle \{v\}, \emptyset \rangle$ is a *minimal prime ideal*. Moreover, every minimal prime ideal is of this form.

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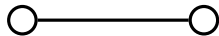
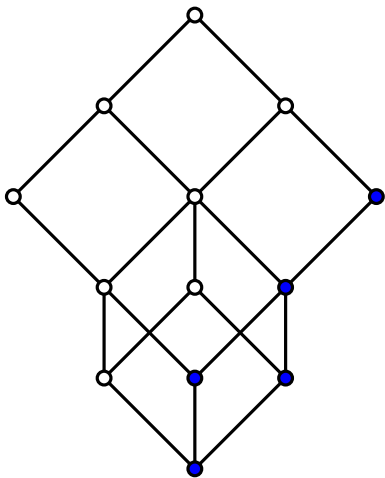
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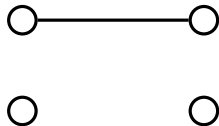
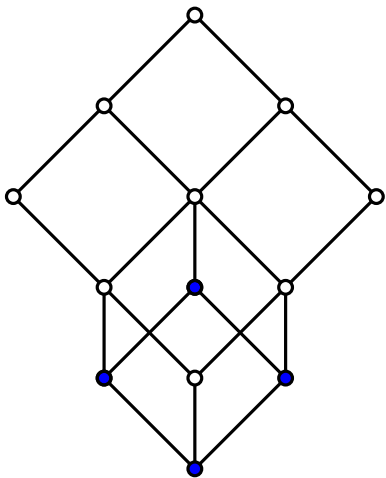
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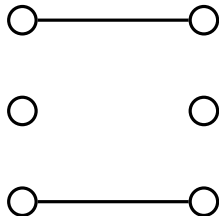
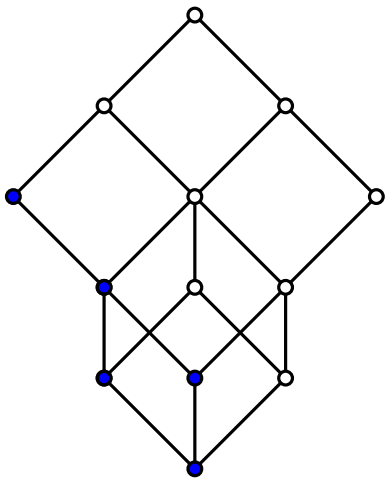
Proposition

Let G be a finite graph. For every vertex v in G , the set $\downarrow \neg \langle \{v\}, \emptyset \rangle$ is a *minimal* prime ideal. Moreover, every minimal prime ideal is of this form.

- Let the deck of a lattice be the set of its minimal prime ideals and define reconstructions analogously.







Translated into the world of lattices

Lattice Reconstruction Problem

Let \mathcal{K} be the class of all lattices \mathbf{L} such that:

- 1 \mathbf{L} is a finite distributive lattice.
- 2 Every prime ideal of \mathbf{L} is minimal or maximal.
- 3 Every join-irreducible element of \mathbf{L} is either an atom or has exactly two atoms below it.

If two lattices in \mathcal{K} are reconstructions of each other, then are they isomorphic?

∖(∩)∖/