Reconstruction of groupoids from pairs of algebras

Aidan Sims
University of Wollongong

Australian Algebra Conference

University of Technology Sydney,
November 28, 2017

UNIVERSITY
OF WOLLONGONG
AUSTRALIA

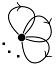
Graphs, algebras, and intriguing coincidences

- A *directed graph* is a collection of dots called vertices connected by arrows called edges.

Graphs, algebras, and intriguing coincidences

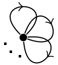
- A *directed graph* is a collection of dots called vertices connected by arrows called edges.
- *Graph C^*-algebras* $C^*(E)$ introduced [KPRR] in late '90's; intensively studied since.

Graphs, algebras, and intriguing coincidences

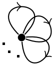
- A *directed graph* is a collection of dots called vertices connected by arrows called edges.
- *Graph C^*-algebras* $C^*(E)$ introduced [KPRR] in late '90's; intensively studied since.
- For the bouquet of n loops  , relations are $S_i^* S_i = 1 = \sum_{j=1}^{n} S_j S_j^*$.

UNIVERSITY
OF WOLLONGONG
AUSTRALIA

Graphs, algebras, and intriguing coincidences

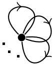
- A *directed graph* is a collection of dots called vertices connected by arrows called edges.
- *Graph C^*-algebras* $C^*(E)$ introduced [KPRR] in late '90's; intensively studied since.
- For the bouquet of n loops  , relations are $S_i^* S_i = 1 = \sum_{j=1}^{n} S_j S_j^*$. Looks like the Leavitt algebra $L_{1,n}$.

Graphs, algebras, and intriguing coincidences

- A *directed graph* is a collection of dots called vertices connected by arrows called edges.
- *Graph C^*-algebras* $C^*(E)$ introduced [KPRR] in late '90's; intensively studied since.
- For the bouquet of n loops  , relations are $S_i^* S_i = 1 = \sum_{j=1}^{n} S_j S_j^*$. Looks like the Leavitt algebra $L_{1,n}$.
- *Leavitt path algebras* $L_R(E)$ are algebraic analogues of graph C^*-algebras, developed in 2004.

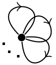
UNIVERSITY OF WOLLONGONG AUSTRALIA

Graphs, algebras, and intriguing coincidences

- A *directed graph* is a collection of dots called vertices connected by arrows called edges.
- *Graph C^*-algebras* $C^*(E)$ introduced [KPRR] in late '90's; intensively studied since.
- For the bouquet of n loops  , relations are $S_i^* S_i = 1 = \sum_{j=1}^{n} S_j S_j^*$. Looks like the Leavitt algebra $L_{1,n}$.
- *Leavitt path algebras* $L_R(E)$ are algebraic analogues of graph C^*-algebras, developed in 2004.
- Remarkably, many theorems from C^*-algebras have directly analogous statements for Leavitt path algebras, and vice versa—but with fundamentally different proofs.

UNIVERSITY OF WOLLONGONG AUSTRALIA

Graphs, algebras, and intriguing coincidences

- A *directed graph* is a collection of dots called vertices connected by arrows called edges.
- *Graph C^*-algebras* $C^*(E)$ introduced [KPRR] in late '90's; intensively studied since.
- For the bouquet of n loops  , relations are $S_i^* S_i = 1 = \sum_{j=1}^{n} S_j S_j^*$. Looks like the Leavitt algebra $L_{1,n}$.
- *Leavitt path algebras* $L_R(E)$ are algebraic analogues of graph C^*-algebras, developed in 2004.
- Remarkably, many theorems from C^*-algebras have directly analogous statements for Leavitt path algebras, and vice versa—but with fundamentally different proofs.
- Key question [AT]: if $C^*(E) \cong C^*(F)$, do we have $L_R(E) \cong L_R(F)$? What about the reverse?

UNIVERSITY OF WOLLONGONG AUSTRALIA

Groupoids

- A groupoid is:
  - "a group with an identity crisis", or

Groupoids

- A groupoid is:
  - "a group with an identity crisis", or
  - a small category with inverses, or

Groupoids

- A groupoid is:
  - “a group with an identity crisis”, or
  - a small category with inverses, or
  - a set with a partially-defined associative multiplication admitting inverses.

Groupoids

- A groupoid is:
  - "a group with an identity crisis", or
  - a small category with inverses, or
  - a set with a partially-defined associative multiplication admitting inverses.
- In a topological groupoid, multiplication and inversion are continuous.

Groupoids

- A groupoid is:
  - "a group with an identity crisis", or
  - a small category with inverses, or
  - a set with a partially-defined associative multiplication admitting inverses.
- In a topological groupoid, multiplication and inversion are continuous.
- Our groupoids are:
  - étale: $s : \gamma \mapsto \gamma^{-1}\gamma$ and $r : \gamma \mapsto \gamma\gamma^{-1}$ are local homeomorphisms (think discrete group); and

UNIVERSITY
OF WOLLONGONG
AUSTRALIA

Groupoids

- A groupoid is:
  - "a group with an identity crisis", or
  - a small category with inverses, or
  - a set with a partially-defined associative multiplication admitting inverses.
- In a topological groupoid, multiplication and inversion are continuous.
- Our groupoids are:
  - étale: $s : \gamma \mapsto \gamma^{-1}\gamma$ and $r : \gamma \mapsto \gamma\gamma^{-1}$ are local homeomorphisms (think discrete group); and
  - ample: the topology is totally disconnected.

Groupoid examples

- Example: $\mathcal{R}_n := \{(i,j) : 1 \leq i,j \leq n\}$; can multiply $(i,j)(k,l)$ if $j = k$, giving (i,l).

Groupoid examples

- Example: $\mathcal{R}_n := \{(i,j) : 1 \leq i,j \leq n\}$; can multiply $(i,j)(k,l)$ if $j = k$, giving (i,l).
- Example: Let E be a directed graph.

Groupoid examples

- Example: $\mathcal{R}_n := \{(i,j) : 1 \leq i,j \leq n\}$; can multiply $(i,j)(k,l)$ if $j = k$, giving (i,l).
- Example: Let E be a directed graph.
  - $E^\infty = \{\text{right-infinite paths } e_1 e_2 e_3 ...\}$.

Groupoid examples

- Example: $\mathcal{R}_n := \{(i,j) : 1 \leq i,j \leq n\}$; can multiply $(i,j)(k,l)$ if $j = k$, giving (i,l).
- Example: Let E be a directed graph.
  - $E^\infty = \{$right-infinite paths $e_1 e_2 e_3 \ldots\}$.
  - $G_E = \{(\alpha x, |\alpha| - |\beta|, \beta x) : \alpha, \beta$ are finite paths and x is an infinite path$\} \subseteq E^\infty \times \mathbb{Z} \times E^\infty$.

Groupoid examples

- Example: $\mathcal{R}_n := \{(i,j) : 1 \leq i,j \leq n\}$; can multiply $(i,j)(k,l)$ if $j = k$, giving (i,l).
- Example: Let E be a directed graph.
  - $E^\infty = \{\text{right-infinite paths } e_1 e_2 e_3 \ldots\}$.
  - $G_E = \{(\alpha x, |\alpha| - |\beta|, \beta x) : \alpha, \beta$ are finite paths and x is an infinite path$\} \subseteq E^\infty \times \mathbb{Z} \times E^\infty$.
  - Can multiply $(w,m,x)(y,n,z)$ if $x = y$ to get $(w, m+n, z)$.

Groupoid examples

- Example: $\mathcal{R}_n := \{(i,j) : 1 \leq i,j \leq n\}$; can multiply $(i,j)(k,l)$ if $j = k$, giving (i,l).
- Example: Let E be a directed graph.
  - $E^\infty = \{\text{right-infinite paths } e_1 e_2 e_3 \dots\}$.
  - $G_E = \{(\alpha x, |\alpha| - |\beta|, \beta x) : \alpha, \beta$ are finite paths and x is an infinite path$\} \subseteq E^\infty \times \mathbb{Z} \times E^\infty$.
  - Can multiply $(w,m,x)(y,n,z)$ if $x = y$ to get $(w, m+n, z)$.
  - Inverse is $(x,m,y)^{-1} = (y,-m,x)$.

Groupoid examples

- Example: $\mathcal{R}_n := \{(i,j) : 1 \le i,j \le n\}$; can multiply $(i,j)(k,l)$ if $j = k$, giving (i,l).
- Example: Let E be a directed graph.
  - $E^\infty = \{$right-infinite paths $e_1 e_2 e_3 \ldots\}$.
  - $G_E = \{(\alpha x, |\alpha| - |\beta|, \beta x) : \alpha, \beta$ are finite paths and x is an infinite path$\} \subseteq E^\infty \times \mathbb{Z} \times E^\infty$.
  - Can multiply $(w,m,x)(y,n,z)$ if $x = y$ to get $(w, m+n, z)$.
  - Inverse is $(x,m,y)^{-1} = (y,-m,x)$.
  - Cylinder sets on E^∞ induce ample étale topology.

Groupoid examples

- Example: $\mathcal{R}_n := \{(i,j) : 1 \leq i,j \leq n\}$; can multiply $(i,j)(k,l)$ if $j = k$, giving (i,l).
- Example: Let E be a directed graph.
  - $E^\infty = \{\text{right-infinite paths } e_1 e_2 e_3 \ldots\}$.
  - $G_E = \{(\alpha x, |\alpha| - |\beta|, \beta x) : \alpha, \beta$ are finite paths and x is an infinite path$\} \subseteq E^\infty \times \mathbb{Z} \times E^\infty$.
  - Can multiply $(w,m,x)(y,n,z)$ if $x = y$ to get $(w, m+n, z)$.
  - Inverse is $(x,m,y)^{-1} = (y, -m, x)$.
  - Cylinder sets on E^∞ induce ample étale topology.
  - This is the *graph groupoid* [KPRR].

UNIVERSITY OF WOLLONGONG AUSTRALIA

Groupoid C^*-algebras and Steinberg algebras

Fix an étale ample groupoid G.

- $C_c(G, \mathbb{C})$ admits a convolution $(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$

UNIVERSITY
OF WOLLONGONG
AUSTRALIA

Groupoid C^*-algebras and Steinberg algebras

Fix an étale ample groupoid G.

- $C_c(G, \mathbb{C})$ admits a convolution $(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$ an an involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$.

Groupoid C^*-algebras and Steinberg algebras

Fix an étale ample groupoid G.

- $C_c(G, \mathbb{C})$ admits a convolution $(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$ an an involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$.
- Complete [R1] to get the groupoid C^*-algebra $C^*(G)$.

UNIVERSITY
OF WOLLONGONG
AUSTRALIA

Groupoid C^*-algebras and Steinberg algebras

Fix an étale ample groupoid G.

- $C_c(G, \mathbb{C})$ admits a convolution $(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$ an an involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$.
- Complete [R1] to get the groupoid C^*-algebra $C^*(G)$.
- Example: $C^*(\mathcal{R}_n) = M_n(\mathbb{C})$.

Groupoid C^*-algebras and Steinberg algebras

Fix an étale ample groupoid G.

- $C_c(G, \mathbb{C})$ admits a convolution $(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$ an an involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$.
- Complete [R1] to get the groupoid C^*-algebra $C^*(G)$.
- Example: $C^*(\mathcal{R}_n) = M_n(\mathbb{C})$.
- Example: $C^*(G_E) = C^*(E)$.

UNIVERSITY OF WOLLONGONG AUSTRALIA

Groupoid C^*-algebras and Steinberg algebras

Fix an étale ample groupoid G.

- $C_c(G, \mathbb{C})$ admits a convolution $(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$ an an involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$.
- Complete [R1] to get the groupoid C^*-algebra $C^*(G)$.
- Example: $C^*(\mathcal{R}_n) = M_n(\mathbb{C})$.
- Example: $C^*(G_E) = C^*(E)$.

There is an algebraic analogue [S, CFST]: fix a commutative ring R.

Groupoid C^*-algebras and Steinberg algebras

Fix an étale ample groupoid G.

- $C_c(G, \mathbb{C})$ admits a convolution $(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$ an an involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$.
- Complete [R1] to get the groupoid C^*-algebra $C^*(G)$.
- Example: $C^*(\mathcal{R}_n) = M_n(\mathbb{C})$.
- Example: $C^*(G_E) = C^*(E)$.

There is an algebraic analogue [S, CFST]: fix a commutative ring R.

- $C_{lc}(G, R)$ (locally constant functions) is an R-algebra (same convolution product).

Groupoid C^*-algebras and Steinberg algebras

Fix an étale ample groupoid G.

- $C_c(G, \mathbb{C})$ admits a convolution $(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$ an an involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$.
- Complete [R1] to get the groupoid C^*-algebra $C^*(G)$.
- Example: $C^*(\mathcal{R}_n) = M_n(\mathbb{C})$.
- Example: $C^*(G_E) = C^*(E)$.

There is an algebraic analogue [S, CFST]: fix a commutative ring R.

- $C_{lc}(G, R)$ (locally constant functions) is an R-algebra (same convolution product).
- This is called the *Steinberg algebra* $A_R(G)$.

Groupoid C^*-algebras and Steinberg algebras

Fix an étale ample groupoid G.

- $C_c(G, \mathbb{C})$ admits a convolution $(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$ an an involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$.
- Complete [R1] to get the groupoid C^*-algebra $C^*(G)$.
- Example: $C^*(\mathcal{R}_n) = M_n(\mathbb{C})$.
- Example: $C^*(G_E) = C^*(E)$.

There is an algebraic analogue [S, CFST]: fix a commutative ring R.

- $C_{lc}(G, R)$ (locally constant functions) is an R-algebra (same convolution product).
- This is called the *Steinberg algebra* $A_R(G)$.
- Example: $A_R(\mathcal{R}_n) = M_n(R)$.

Groupoid C^*-algebras and Steinberg algebras

Fix an étale ample groupoid G.

- $C_c(G, \mathbb{C})$ admits a convolution $(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$ an an involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$.
- Complete [R1] to get the groupoid C^*-algebra $C^*(G)$.
- Example: $C^*(\mathcal{R}_n) = M_n(\mathbb{C})$.
- Example: $C^*(G_E) = C^*(E)$.

There is an algebraic analogue [S, CFST]: fix a commutative ring R.

- $C_{lc}(G, R)$ (locally constant functions) is an R-algebra (same convolution product).
- This is called the *Steinberg algebra* $A_R(G)$.
- Example: $A_R(\mathcal{R}_n) = M_n(R)$.
- Example: $A_R(G_E) \cong L_R(E)$ [CFST].

UNIVERSITY OF WOLLONGONG AUSTRALIA

No coincidence after all

A lot begins to come clear. Example (one of many):

Theorem. [BPRS, AA-P] Let E be a directed graph and R a simple ring. TFAE: (1) E is cofinal and every cycle has an exit; (2) $C^*(E)$ is simple; and (3) $L_R(E)$ is simple.

No coincidence after all

A lot begins to come clear. Example (one of many):

Theorem. [BPRS, AA-P] Let E be a directed graph and R a simple ring. TFAE: (1) E is cofinal and every cycle has an exit; (2) $C^*(E)$ is simple; and (3) $L_R(E)$ is simple.

Proof 1. Fundamentally different arguments show that each of conditions (2) and (3) is equivalent to condition (1).

No coincidence after all

A lot begins to come clear. Example (one of many):

Theorem. [BPRS, AA-P] Let E be a directed graph and R a simple ring. TFAE: (1) E is cofinal and every cycle has an exit; (2) $C^*(E)$ is simple; and (3) $L_R(E)$ is simple.

Proof 1. Fundamentally different arguments show that each of conditions (2) and (3) is equivalent to condition (1).

Proof 2. [BCFS] If G is an ample étale groupoid[1], then very closely related arguments prove that $A_R(G)$ and $C^*(G)$ are simple if and only if G is *topologically principal* and *minimal*.

---

[1]Warning: not suitable for use on nonamenable groupoids. Terms and conditions apply.

No coincidence after all

A lot begins to come clear. Example (one of many):

Theorem. [BPRS, AA-P] Let E be a directed graph and R a simple ring. TFAE: (1) E is cofinal and every cycle has an exit; (2) $C^*(E)$ is simple; and (3) $L_R(E)$ is simple.

Proof 1. Fundamentally different arguments show that each of conditions (2) and (3) is equivalent to condition (1).

Proof 2. [BCFS] If G is an ample étale groupoid[1], then very closely related arguments prove that $A_R(G)$ and $C^*(G)$ are simple if and only if G is *topologically principal* and *minimal*.

Can groupoids help with the isomorphism question?

[1]Warning: not suitable for use on nonamenable groupoids. Terms and conditions apply.

Renault's reconstruction theorem

Background: if G is étale, then $G^0 := \{\gamma\gamma^{-1} : \gamma \in G\}$ is clopen.
So $D := C_0(G^0)$ is a commutative C^*-subalgebra of $C^*(G)$.

Renault's reconstruction theorem

Background: if G is étale, then $G^0 := \{\gamma\gamma^{-1} : \gamma \in G\}$ is clopen. So $D := C_0(G^0)$ is a commutative C^*-subalgebra of $C^*(G)$.

Theorem. [R2] If G is topologically principal, then we can recover G from $(C^*(G), D)$.

Renault's reconstruction theorem

Background: if G is étale, then $G^0 := \{\gamma\gamma^{-1} : \gamma \in G\}$ is clopen. So $D := C_0(G^0)$ is a commutative C^*-subalgebra of $C^*(G)$.

Theorem. [R2] If G is topologically principal, then we can recover G from $(C^*(G), D)$.

Proof. • Gelfand duality recovers G^0 from $C_0(G^0)$.

Renault's reconstruction theorem

Background: if G is étale, then $G^0 := \{\gamma\gamma^{-1} : \gamma \in G\}$ is clopen. So $D := C_0(G^0)$ is a commutative C^*-subalgebra of $C^*(G)$.

Theorem. [R2] If G is topologically principal, then we can recover G from $(C^*(G), D)$.

Proof. • Gelfand duality recovers G^0 from $C_0(G^0)$.
• Let $N(D) := \{n \in C^*(G) : nDn^*, n^*Dn \subseteq D\}$.

UNIVERSITY OF WOLLONGONG AUSTRALIA

Renault's reconstruction theorem

Background: if G is étale, then $G^0 := \{\gamma\gamma^{-1} : \gamma \in G\}$ is clopen. So $D := C_0(G^0)$ is a commutative C^*-subalgebra of $C^*(G)$.

Theorem. [R2] If G is topologically principal, then we can recover G from $(C^*(G), D)$.

Proof. • Gelfand duality recovers G^0 from $C_0(G^0)$.
• Let $N(D) := \{n \in C^*(G) : nDn^*, n^*Dn \subseteq D\}$.
• Polar decomposition of n in $C^*(G)^{**}$ determines homeomorphism $\alpha_n : \{x : n^*n(x) > 0\} \to \{x : nn^*(x) > 0\}$.

Renault's reconstruction theorem

Background: if G is étale, then $G^0 := \{\gamma\gamma^{-1} : \gamma \in G\}$ is clopen. So $D := C_0(G^0)$ is a commutative C^*-subalgebra of $C^*(G)$.

Theorem. [R2] If G is topologically principal, then we can recover G from $(C^*(G), D)$.

Proof. • Gelfand duality recovers G^0 from $C_0(G^0)$.
• Let $N(D) := \{n \in C^*(G) : nDn^*, n^*Dn \subseteq D\}$.
• Polar decomposition of n in $C^*(G)^{**}$ determines homeomorphism $\alpha_n : \{x : n^*n(x) > 0\} \to \{x : nn^*(x) > 0\}$.
• On $\{(n, x) : n^*n(x) > 0\}$, define $[n, x] \sim [m, y]$ if $x = y$ and $\alpha_n = \alpha_m$ on a neighbourhood of x.

Renault's reconstruction theorem

Background: if G is étale, then $G^0 := \{\gamma\gamma^{-1} : \gamma \in G\}$ is clopen. So $D := C_0(G^0)$ is a commutative C^*-subalgebra of $C^*(G)$.

Theorem. [R2] If G is topologically principal, then we can recover G from $(C^*(G), D)$.

Proof. • Gelfand duality recovers G^0 from $C_0(G^0)$.
• Let $N(D) := \{n \in C^*(G) : nDn^*, n^*Dn \subseteq D\}$.
• Polar decomposition of n in $C^*(G)^{**}$ determines homeomorphism $\alpha_n : \{x : n^*n(x) > 0\} \to \{x : nn^*(x) > 0\}$.
• On $\{(n, x) : n^*n(x) > 0\}$, define $[n, x] \sim [m, y]$ if $x = y$ and $\alpha_n = \alpha_m$ on a neighbourhood of x.
• Now $G \cong \{[n, x] : n^*n(x) > 0\}$: for $\gamma \in G$, fix $f \in C_c(G)$ with r, s injective on $\text{supp}(f)$, and then $\gamma \mapsto [f, s(\gamma)]$.

Reconstruction from Steinberg algebras

If G is étale and ample, and R is a commutative ring, then $D_0 := C_{lc}(G^0, R)$ is a commutative subring of $A_R(G)$.

Reconstruction from Steinberg algebras

If G is étale and ample, and R is a commutative ring, then $D_0 := C_{lc}(G^0, R)$ is a commutative subring of $A_R(G)$.

Theorem. [ABHS (see also [BCaH])] If G is topologically principal and R is a commutative integral domain with 1, then we can recover G from $(A_R(G), D)$.

UNIVERSITY
OF WOLLONGONG
AUSTRALIA

Reconstruction from Steinberg algebras

If G is étale and ample, and R is a commutative ring, then $D_0 := C_{lc}(G^0, R)$ is a commutative subring of $A_R(G)$.

Theorem. [ABHS (see also [BCaH])] If G is topologically principal and R is a commutative integral domain with 1, then we can recover G from $(A_R(G), D)$.

Proof. Same idea as before but:

Reconstruction from Steinberg algebras

If G is étale and ample, and R is a commutative ring, then $D_0 := C_{lc}(G^0, R)$ is a commutative subring of $A_R(G)$.

Theorem. [ABHS (see also [BCaH])] If G is topologically principal and R is a commutative integral domain with 1, then we can recover G from $(A_R(G), D)$.

Proof. Same idea as before but:
• no involution to define normalisers; instead $n \in N(D)$ if $\exists m$ with $nmn = n$, $mnm = m$ and $nDm, mDn \subseteq D$.

UNIVERSITY
OF WOLLONGONG
AUSTRALIA

Reconstruction from Steinberg algebras

If G is étale and ample, and R is a commutative ring, then $D_0 := C_{lc}(G^0, R)$ is a commutative subring of $A_R(G)$.

Theorem. [ABHS (see also [BCaH])] If G is topologically principal and R is a commutative integral domain with 1, then we can recover G from $(A_R(G), D)$.

Proof. Same idea as before but:
• no involution to define normalisers; instead $n \in N(D)$ if $\exists m$ with $nmn = n$, $mnm = m$ and $nDm, mDn \subseteq D$.
• no Gelfand duality, but we use Stone duality: integral domain ensures no extraneous idempotents.

Reconstruction from Steinberg algebras

If G is étale and ample, and R is a commutative ring, then $D_0 := C_{lc}(G^0, R)$ is a commutative subring of $A_R(G)$.

Theorem. [ABHS (see also [BCaH])] If G is topologically principal and R is a commutative integral domain with 1, then we can recover G from $(A_R(G), D)$.

Proof. Same idea as before but:
• no involution to define normalisers; instead $n \in N(D)$ if $\exists m$ with $nmn = n$, $mnm = m$ and $nDm, mDn \subseteq D$.
• no Gelfand duality, but we use Stone duality: integral domain ensures no extraneous idempotents.
• no polar decomposition or double-dual;

Reconstruction from Steinberg algebras

If G is étale and ample, and R is a commutative ring, then $D_0 := C_{lc}(G^0, R)$ is a commutative subring of $A_R(G)$.

Theorem. [ABHS (see also [BCaH])] If G is topologically principal and R is a commutative integral domain with 1, then we can recover G from $(A_R(G), D)$.

Proof. Same idea as before but:
• no involution to define normalisers; instead $n \in N(D)$ if $\exists m$ with $nmn = n$, $mnm = m$ and $nDm, mDn \subseteq D$.
• no Gelfand duality, but we use Stone duality: integral domain ensures no extraneous idempotents.
• no polar decomposition or double-dual; but a "standard form" for normalisers shows that m is unique given n, and mpn is idempotent when $p \in D$ is idempotent.

Reconstruction from Steinberg algebras

If G is étale and ample, and R is a commutative ring, then $D_0 := C_{lc}(G^0, R)$ is a commutative subring of $A_R(G)$.

Theorem. [ABHS (see also [BCaH])] If G is topologically principal and R is a commutative integral domain with 1, then we can recover G from $(A_R(G), D)$.

Proof. Same idea as before but:
• no involution to define normalisers; instead $n \in N(D)$ if $\exists m$ with $nmn = n$, $mnm = m$ and $nDm, mDn \subseteq D$.
• no Gelfand duality, but we use Stone duality: integral domain ensures no extraneous idempotents.
• no polar decomposition or double-dual; but a "standard form" for normalisers shows that m is unique given n, and mpn is idempotent when $p \in D$ is idempotent. So we obtain α_n on the Stone spectrum.

Diagonal-preserving isomorphism

We obtain some progress on the isomorphism question:

Theorem. [ABHS (see also [BCaH])] If E and F are graphs in which every cycle has an exit then the following are equivalent:
• there is a diagonal-preserving isomorphism $C^*(E) \cong C^*(F)$;
• there exists a commutative integral domain R with 1 for which there is a diagonal-preserving isomorphism $L_R(E) \cong L_R(F)$;
• for every ring R, there is a diagonal-preserving isomorphism $L_R(E) \cong L_R(F)$;
• the groupoids G_E and G_F are isomorphic.

References

[AT] G. Abrams and M. Tomforde, *Isomorphism and Morita equivalence of graph algebras*, Trans. Amer. Math. Soc. **363** (2011), 3733–3767.

[AA-P] G. Abrams and G. Aranda Pino, *The Leavitt path algebra of a graph*, J. Algebra **293** (2005), 319–334.

[BPRS] T. Bates, D. Pask, I. Raeburn and W. Szymański, *The C^*-algebras of row-finite graphs*, New York J. Math. **6** (2000), 307–324.

[BCFS] J. Brown, L.O. Clark, C. Farthing and A. Sims, *Simplicity of algebras associated to étale groupoids*, Semigroup Forum **88** (2014), 433–452.

[BCaH] J. Brown, L.O. Clark and A. an Huef, *Diagonal-preserving ring *-isomorphisms of Leavitt path algebras*, J. Pure Appl. Algebra **221** (2017), 2458–2481.

[CFST] L.O. Clark, C. Farthing, A. Sims and M. Tomforde, *A groupoid generalisation of Leavitt path algebras*, Semigroup Forum **89** (2014), 501–517.

[KPRR] A. Kumjian, D. Pask, I. Raeburn and J. Renault, *Graphs, groupoids, and Cuntz-Krieger algebras*, J. Funct. Anal. **144** (1997), 505–541.

[R1] J. Renault, A groupoid approach to C^*-algebras, Springer, Berlin, 1980, ii+160.

[R2] J. Renault, *Cartan subalgebras in C^*-algebras*, Irish Math. Soc. Bulletin **61** (2008), 29–63.

[S] B. Steinberg, *A groupoid approach to discrete inverse semigroup algebras*, Adv. Math. **223** (2010), 689–727.