Separability and commensurated subgroups

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Let G be a group. Two subgroups H and K of G are **commensurate** if $|H : H \cap K|$ and $|K : H \cap K|$ are finite.
The **commensurator** of H in G is the group $\mathrm{Comm}_G(H)$ of elements $g \in G$ such that gHg^{-1} is commensurate with H.
H is **commensurated** in G if $\mathrm{Comm}_G(H) = G$.

'Trivial' examples:

- Normal subgroups are commensurated.
- If H is commensurated, and K is commensurate with H, then K is commensurated. In particular, **virtually normal** subgroups are commensurated, that is, subgroups H of G such that some finite index subgroup of H is normal in G.

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More interesting sources of examples (not virtually normal in general):

- Examples arising from arithmetic groups, e.g. $\mathrm{SL}_n(\mathbb{Z})$ is commensurated in $\mathrm{SL}_n(\mathbb{Q})$.
- If X is a proper metric space and $G \leq \mathrm{Isom}(X)$ is such that orbits of G are closed and discrete, then the stabilizers in G of points in X are all commensurate to one another; in particular, they are commensurated in G.
- If G is a totally disconnected locally compact (t.d.l.c.) group, then G has a base of identity neighbourhoods consisting of compact open subgroups (Van Dantzig's theorem). All such subgroups are commensurate to one another, so in particular they are commensurated.

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The **profinite topology** of the group G is the coarsest group topology in which every finite index subgroup is open. The closure of $H \leq G$ in this topology is given by

$$\overline{H} = \bigcap_{K \in \mathcal{K}_H} K = \bigcap_{N \in \mathcal{N}} HN,$$

where $\mathcal{K}_H$ is the set of finite index subgroups containing H and $\mathcal{N}$ is the set of finite index normal subgroups of G. If $H = \overline{H}$ we say H is **separable** in G. There is always a smallest separable subgroup, formed as the intersection of all (normal) subgroups of finite index; groups in which the trivial subgroup is separable are known as **residually finite** groups.

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In general it is hard to tell whether a given subgroup is separable (even if G is residually finite). For example, a non-abelian free group G has many infinite simple quotients G/H, and then K is inseparable in G for all $H \leq K < G$.

We define the **residual closure** of $H \leq G$ to be

$$\widetilde{H} = \bigcap_{N \in \mathcal{N}_H} HN,$$

where now $\mathcal{N}_H$ is the set of normal subgroups of G such that $|HN : N| < \infty$. H is **weakly separable** if $H = \widetilde{H}$; equivalently, H is an intersection of virtually normal subgroups.

In general $\widetilde{H} \leq \overline{H}$; think of weakly separable as a common generalization of 'separable' and 'normal'.

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Main Theorem (Caprace–Kropholler–R.–Wesolek)

Let G be a group and let H be a commensurated subgroup of G. Suppose that G is generated by finitely many cosets of H. Then $\widetilde{H}$ is virtually normal in G, that is,

$$N = \bigcap_{g \in G} g\widetilde{H}g^{-1}$$

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Corollary

Let G be a finitely generated group. Then every commensurated weakly separable subgroup of G is virtually normal.

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From now on 'Corollary' = corollary/application of the Main Theorem.

The **Hirsch length** $h(G)$ of a soluble group is

$$h(G) = \sum_{j \geq 0} \dim G^{(j)}/G^{(j+1)} \otimes \mathbb{Q};$$

this generalizes to virtually soluble groups by setting $h(G) = h(H)$ for some (any) finite index soluble subgroup H.
A group G is **polycyclic** if it has a series

$$\{1\} \lhd G_1 \lhd G_2 \cdots \lhd G_n = G$$

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We obtain several characterizations of virtually polycylic groups in terms of separability of subgroups.

Corollary (based on discussions between P. Kropholler and B. Nucinkis; cf Jeanes–WIlson)

Let G be a finitely generated virtually soluble group. Then the following are equivalent:

(i) G is virtually polycyclic.

(ii) Every subgroup of G is separable.

(iii) Every subnormal subgroup of G is separable.

(iv) For all $H \leq G$, H is virtually normal in $\mathrm{Comm}_G(H)$.

(v) For all subnormal subgroups H of G and all $J \subseteq \mathrm{Comm}_G(H)$ finite, H is virtually normal in $\langle J \cup H \rangle$.

As shown by Meskin, most Baumslag–Solitar groups (groups of the form $\mathbb{Z}*_\phi$) are not residually finite. More generally, consider the class $\mathcal{P}_n$ of fundamental groups of graphs of groups, such that each vertex and edge group is virtually polycyclic of some fixed Hirsch length n. For example, generalized Baumslag–Solitar groups belong to $\mathcal{P}_1$.

We find that residually finite groups in $\mathcal{P}_n$ have a special structure.

Corollary

Suppose $G \in \mathcal{P}_n$ is residually finite and does not fix any end of the associated Bass–Serre tree T. Then G has a polycyclic normal subgroup N of Hirsch length n, such that N fixes a vertex of T and G/N is virtually free.

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Corollary

Let X be a uniquely geodesic metric space such that closed balls in X are compact, and let $G \leq \mathrm{Isom}(X)$ be finitely generated. Suppose that G does not preserve any proper nonempty convex subspace of X, and suppose there is $x \in X$ such that $\overline{Gx}$ is discrete. Then G_x is commensurated and the following are equivalent:

(i) G_x is weakly separable in G;

(ii) G_x is finite;

(iii) G is **properly discontinuous**: for each compact subset K of X, the set $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite.

A **lattice** in a locally compact group G is a discrete subgroup Γ, such that G/Γ admits a finite invariant measure. Lattices are a powerful tool for understanding semisimple algebraic groups, and more generally, lattices in locally compact groups often give interesting examples of properly discontinuous groups of isometries, and vice versa. If G is compactly generated and G/Γ is compact, then Γ is finitely generated.

For general locally compact groups, lattice theory is most well-developed in the following situation: Γ is an **irreducible lattice** in a direct product $G_1 \times G_2$ of locally compact groups, meaning that the projection of Γ onto each factor is nondiscrete. The best-known non-linear examples (due to e.g. Burger–Mozes) come from the case when G_1 and G_2 are automorphism groups of locally finite trees.

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Write pr_i for the projection map from Γ to G_i.

Corollary (cf Caprace–Monod)

Suppose that Γ is a finitely generated, residually finite lattice in the product $G_1 \times G_2$ of locally compact groups. Suppose that G_1 is a nondiscrete t.d.l.c. group, every infinite closed normal subgroup of G_1 has trivial centralizer, and that $\mathrm{pr}_1(\Gamma)$ is dense in G_1. Then pr_2 is injective.

Corollary (cf Burger–Mozes)

Let T_1 and T_2 be locally finite leafless trees. Suppose that $\Gamma \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ has finite point stabilizers on $T_1 \times T_2$ and that $\Gamma \backslash (T_1 \times T_2)$ is compact. Then the following are equivalent:

(i) There exists $i \in \{1, 2\}$ and $v \in VT_i$ such that Γ_v is weakly separable in Γ;

(ii) $|\Gamma : \Gamma_1 \times \Gamma_2| < \infty$, where $\Gamma_i = \Gamma \cap \mathrm{Aut}(T_i)$.

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