Gradings in algebra and analysis

Iain Raeburn

Victoria University of Wellington

Let $E = (E^0, E^1, r, s)$ be a directed graph. A *Cuntz-Krieger E-family* in a C^*-algebra A (think $B(H)$) consists of mutually orthogonal projections $\{P_v : v \in E^0\} \subset A$ and $\{S_e : e \in E^1\} \subset A$ satisfying

(CK1) $S_e^* S_e = P_{s(e)}$ for $e \in E^1$, and

(CK2) $P_v = \sum_{r(e)=v} S_e S_e^*$ for $v \in E^0$.

The *graph C^*-algebra* $C^*(E)$ is generated by a universal Cuntz-Krieger E-family $\{p_v\} \cup \{s_e\}$.

Sample Theorem (Kumjian-Pask-R 1998). Suppose that every cycle in E has an entry, and that E is *cofinal*: for every infinite path $\mu = \mu_1 \mu_2 \cdots$ and every $v \in E^0$, $vE^* s(\mu_i) \neq \emptyset$ for some i. Then $C^*(E)$ is simple.

Let K be a field. The *Leavitt path algebra* $L_K(E)$ is generated by $E^0 \cup E^1 \cup \left(E^{1*} = \{e^* : e \in E^1\}\right)$ subject to

(V) $vw = \delta_{v,w} v$ for $v, w \in E^0$,

(E1) $es(e) = e = r(e)e$ for $e \in E^1$,

(E2) $e^* r(e) = e^* = s(e)e^*$ for $e \in E^1$,

(CK1) $e^* e = s(e)$ for $e \in E^1$, and

(CK2) $v = \sum_{r(e)=v} ee^*$ for $v \in E^0$.

Sample Theorem (Abrams-Aranda Pino 2005). Suppose that every cycle in E has an entry, and that E is *cofinal*: for every infinite path $\mu = \mu_1 \mu_2 \cdots$ and every $v \in E^0$, $vE^* s(\mu_i) \neq \emptyset$ for some i. Then $L_K(E)$ is simple.

Many theorems about graph C^*-algebras have analogues for Leavitt path algebras. But the proofs are not the same. The difference is that Leavitt path algebras are graded.

Suppose G is a group. An algebra A is *G-graded* if there are subspaces $\{A_g : g \in G\}$ such that $A = \bigoplus_{g \in G} A_g$ and $A_g A_h \subset A_{gh}$. Then each $a \in A$ has a unique expansion $a = \sum a_g$ as a sum of *homogeneous components* $a_g \in A_g$.

In a Leavitt path algebra, we have a copy of the entire path space E^*: just multiply the edges. Then the spaces $L_K(E)_n := \operatorname{span}\{\mu\nu^* : |\mu| - |\nu| = n\}$ give a $\mathbb{Z}$-grading of $L_K(E)$.

So do the closed suspaces $C^*(E)_n = \overline{\operatorname{span}}\{s_\mu s_\nu^* : |\mu| - |\nu| = n\}$ form a grading of $C^*(E)$? No, definitely not.

Example. Consider the graph with one vertex and one edge e. Then $C^*(E) = C^*(S_e)$ is $C(\mathbb{T}) = C^*(z)$. Each $f \in C(\mathbb{T})$ has natural graded components $\widehat{f}(n)z^n$, where $\widehat{f}(n)$ are the Fourier coefficients of f. We can have $\widehat{f}(n) \neq 0$ for infinitely many n, so $\sum \widehat{f}(n)z^n$ is an infinite series. But unless f is smooth, the series need not converge, at least in the norm of $C(\mathbb{T}) = C^*(E)$.

A graph algebra $C^*(E)$ carries a canonical gauge action α of $\mathbb{R}$ such that $\alpha_t(p_v) = p_v$ and $\gamma_t(s_e) = e^{2\pi it}s_e$. Then for $n \in \mathbb{Z}$ and $a \in C^*(E)$, we can define "Fourier coefficients"

$$a_n := \int_0^1 \gamma_t(a)e^{-2\pi int}\,dt \in C^*(E)_n.$$

The algebraic direct sum $\bigoplus C^*(E)_n$ is dense in $C^*(E)$, but can we get a from $\{a_n\}$? The example shows this a delicate question. But:

Claim. (Abrams-Ara-Miles Solina, 2017) "$C^*(E)$ is graded".

Exel (1997): A C^*-algebra is *G-graded* if there are closed, linearly independent subspaces A_g such that $A_gA_h \subset A_{gh}$ and $\bigoplus A_g$ is dense in A. Two issues:

- No way to assign homogeneous components to each $a \in A$; and hence
- there is no mention of recovery of $a \in A$.

Exel (1997): A G-graded C^*-algebra is *topologically G-graded* if the map $a \mapsto a_e$ on $\bigoplus_g A_g$ extends to a bounded linear operator F on A.

In fact Abrams et al do enough to prove that $C^*(E)$ is topologically $\mathbb{Z}$-graded *using the gauge action*. This is not an accident:

Theorem. Suppose that G is an abelian group, and A is a toplogically G-graded C^*-algebra. Then there is a strongly continuous action of the compact dual $\widehat{G}$ on A such that $\alpha(a) = \gamma(g)a$ for $a \in A_g$ and

$$F(a) = \int_{\widehat{G}} \alpha_\gamma(a)\, d\gamma \quad \text{for } a \in A.$$

Exel (2017 book): A is universal for representations of "Fell bundles". This allows us to define automorphisms α_γ. For $C^*(E)$, we recover the gauge action of $\mathbb{T} = \widehat{\mathbb{Z}}$.

The Fourier coefficients need not decay fast enough to make the usual partial sums

$$s_N(f) := \sum_{n=-N}^{N} a_n = \sum_{n=-N}^{N} \widehat{f}(n)e^{2\pi ins}$$

converge uniformly. But (Fejér, 1900): the Césaro sums

$$\sigma_N(f) := \frac{1}{N+1}\sum_{n=0}^{N} s_N(f)$$

of the Fourier series converge uniformly to f.

Theorem. Suppose that α is a continuous action of $\mathbb{T}$ on a C^*-algebra A. For $a \in A$, set

$$a_n := \int_0^1 \alpha_t(a)e^{-2\pi int}\,dt,$$

and take $s_N(a)$, $\sigma_N(a)$ as above. Then $\sigma_N(a) \to a$ as $N \to \infty$.