Stability of Gorenstein Graded Flat Modules

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Background

- Gorenstein homological algebra (Auslander and Bridger, 1969)
- Gorenstein graded-injective and graded-projective modules, Gorensiten graded-flat modules (Asensio, Lopez Ramos, Torecillas, 1998)
- Stability of Gorenstein flat modules (Sather-Wagstraff, Sharif, White, 2011) (Bouchiba and Khaloui, 2012)
- Stability of Gorenstein n-flat modules (Selvaraj and Udhayakumar, 2014)
- Stability of strongly Gorenstein flat modules (Wang and Liu, 2014)
- Strongly Gorenstein graded modules (Mao, 2017)

Aim

Stability of Gorenstein graded flat modules over graded rings.

Basic Definitions

- All rings are associative with identity element and the (left or right) R-modules are unital.
- By R-Mod we denote the Grothendieck category of all left R-modules.
- Let G be a multiplicative group with identity element e. A graded ring R is a ring with identity 1, together with a direct decomposition $R = \oplus_{\sigma \in G} R_\sigma$ (as additive subgroups) such that $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ for all $\sigma, \tau \in G$. Thus R_e is a subring of R, $1 \in R_e$ and for every $\sigma \in G$, R_σ is an R_e-bimodule.
- A left graded R-module is a left R-module M endowed with an internal direct sum decomposition $M = \oplus_{\sigma \in G} M_\sigma$, where each M_σ is a subgroup of the additive group of M such that $R_\sigma M_\tau \subseteq M_{\sigma\tau}$ for all $\sigma, \tau \in G$.

- For M and N graded left R-modules, we put $Hom_{R-gr}(M,N) = \{f : M \to N |$ is R-linear and $f(M_\sigma) \subseteq N_\sigma, \forall \sigma \in G\}$.
- $Hom_{R-gr}(M,N)$ is the group of all morphisms from M to N in the category R-gr of the graded left R-modules (gr-R will denote the category of the graded right R-modules).
- It is well known that R-gr is a Grothendieck category.

- An R-linear $f : M \longrightarrow N$ is said to be a *graded morphism of degree* $\tau, \tau \in G$, if $f(M_\sigma) \subseteq M_{\sigma\tau}$ for all $\sigma \in G$.
- Graded morphisms of degree τ build an additive subgroup $HOM_R(M, N)_\tau$ of $Hom_R(M, N)$.
- It is clear that $HOM_R(M, N)_e = Hom_{R-gr}(M, N)$.
- We will denote Ext_R^i, Ext_{R-gr}^i, and EXT_R^n as the left derived functor of Hom_R, Hom_{R-gr}, and HOM_R, respectively.

- Let M be a graded right R-module and N a graded left R-module. The abelian group $M \otimes_R N$ may be graded by putting $(M \otimes_R N)_\sigma, \sigma \in G$, equal to the additive subgroup generated by elements $x \otimes y$ with $x \in M_\alpha, y \in N_\beta$ such that $\alpha\beta = \sigma$.
- The projective objects of R-*gr* will be called projective (resp. flat) graded modules because M is *gr*-projective (rep. *gr*-flat) if and if only is a projective (resp. flat) module.

Definition (Enochs and Jenda, 2000)

Let R be a ring and let $\mathfrak{X}$ be a class of left R-modules.

(1) $\mathfrak{X}$ *is closed under extensions: If for every short exact sequence of left R-modules $0 \to A \to B \to C \to 0$, the conditions A and C are in $\mathfrak{X}$ implies B is in $\mathfrak{X}$.*

(2) $\mathfrak{X}$ *is closed under kernels of epimorphisms: If for every short exact sequence of left R-modules $0 \to A \to B \to C \to 0$, the conditions B and C are in $\mathfrak{X}$ implies A is in $\mathfrak{X}$.*

(3) $\mathfrak{X}$ *is projectively resolving: If it contains all projective modules and it is closed under both extensions and kernels of epimorphisms, i.e., for every short exact sequence of R-modules $0 \to A \to B \to C \to 0$ with $C \in \mathfrak{X}$, the conditions $A \in \mathfrak{X}$ and $B \in \mathfrak{X}$ are equivalent.*

Definition (Asensio, Lopez Ramos, Torrecillas, 1998)

An exact sequence

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of gr-flat left R-modules in R-gr is called a complete gr-flat resolution if $E \otimes_R -$ *leaves the sequence exact for any gr-injective right R-module E. A graded left R-module M is called Gorenstein gr-flat if there is a complete gr-flat resolution above such that* $M \cong Ker(F^0 \to F^1)$.

Definition

A graded right R-module M is said to be Gorenstein gr-injective, if there exists an exact sequence of gr-injective right R-modules

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

such that $M \cong Ker(E^0 \to E^1)$ *and such that* $Hom_{R-gr}(E, -)$ *leaves the sequence exact whenever E is an gr-injective right R-module.*

Two-degree Gorenstein *gr*-flat modules

Definition

A graded left R-module M is called two-degree Gorenstein gr-flat if there exists an exact sequence of Gorenstein gr-flat left R-modules

$$\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

such that $M \cong Im(G_0 \to G^0)$ *and it remains exact after applying* $H \otimes_R -$ *for any Gorenstein gr-injective right R-module H.*

- Let $\mathcal{GF}_{gr}(R)$, and $\mathcal{GF}^{(2)}_{gr}(R)$ be the class of all Gorenstein *gr*-flat left, two-degree Gorenstein *gr*-flat left modules over R respectively.
- Also denote $\mathcal{GF}^{(2)}_{i-gr}(R)$ the subcategory of R-*gr* for which there exists an exact sequence of Gorenstein *gr*-flat R-modules

$$\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

such that $M \cong Im(G_0 \to G^0)$ and it remains exact after applying $E \otimes_R -$ for any *gr*-injective R-module E. It is routine to check that

$$\mathcal{GF}_{gr}(R) \subseteq \mathcal{GF}^{(2)}_{gr}(R) \subseteq \mathcal{GF}^{(2)}_{i-gr}(R).$$

Main Theorem

Let R be a left GF-gr-closed ring. Then

$$\mathcal{GF}_{gr}(R) = \mathcal{GF}^{(2)}_{gr}(R) = \mathcal{GF}^{(2)}_{i-gr}(R).$$

GF-gr-closed ring

Definition

A ring R is said to be left GF-gr-closed if $\mathcal{GF}_{gr}(R)$ is closed under extensions.

Lemma (1)

The following are equivalent for a graded left R-module M:

(1) *M is Gorenstein gr-flat;*

(2) *M satisfies the two following conditions:*

   (i) *$\mathrm{Tor}_i(E, M) = 0$ for all $i > 0$ and all gr-injective right R-modules E; and*

   (ii) *There exists an exact sequence in R-gr, $0 \to M \to F^0 \to F^1 \to \cdots$, with each F^i is gr-flat, such that $E \otimes_R -$ leaves the sequence exact whenever E is gr-injective right R-module;*

(3) *There exists a short exact sequence in R-gr, $0 \to M \to F \to G \to 0$, where F is gr-flat and G is Gorenstein gr-flat.*

Lemma (2)

Let $0 \to A \to B \to C \to 0$ be a short exact sequence in R-gr. If A is Gorenstein gr-flat and C is gr-flat, then B is Gorenstein gr-flat

Theorem (3)

If R is a left GF-gr-closed ring, then the class $\mathcal{GF}_{gr}(R)$ is closed under direct summands.

Stability of Gorenstein *gr*-flat modules

First, let us call Gorenstein G *gr*-flat module, any element of $\mathcal{GF}^{(2)}_{i-gr}(R)$.

Definition

A graded left R-module M is called a strongly Gorenstein gr-flat module if there exists an exact sequence of R-modules,

$$0 \to M \to F \to M \to 0$$

in R-gr such that F is a gr-flat and $E \otimes_R -$ leaves this sequence exact whenever E is an gr-injective right module.

Definition

A graded left R-module M is called a strongly Gorenstein G gr-flat module if there exists an exact sequence $0 \to M \to G \to M \to 0$ in R-gr such that G is Gorenstein gr-flat over R and $E \otimes_R -$ leaves this sequence exact for each gr-injective right R-module E.

Proposition (1)

1. *Any strongly Gorenstein G gr-flat module is Gorenstein G gr-flat.*
2. *The family of Gorenstein G gr-flat modules is stable under arbitrary direct sums.*

Proposition (2)

Let M be a graded left R-module. Then the following statements are equivalent:

(1) *M is a strongly Gorenstein G gr-flat module.*

(2) *There exists an exact sequence $0 \to M \to G \to M \to 0$ in R-gr such that G is a Gorenstein gr-flat module, and $Tor_1(E, M) = 0$ for any gr-injective right R-module E.*

(3) *There exists an exact sequence $0 \to M \to G \to M \to 0$ in R-gr such that G is a Gorenstein gr-flat module and, for any right gr-injective R-module E, the following sequence is exact*

$$0 \to E \otimes M \to E \otimes G \to E \otimes M \to 0.$$

Proposition (3)

Let R be a graded ring and let M be a Gorenstein G gr-flat R-module. Then M is a direct summand of a strongly Gorenstein G gr-flat module.

For easiness, we adopt the following definition.

Definition

Let M be a strongly Gorenstein G gr-flat module. An R-gr-module N is called an M_{gr}-type module if there exists an exact sequence $0 \to M \to N \to H \to 0$ in R-gr such that H is a Gorenstein gr-flat module.

Proposition (4)

Let M be a strongly Gorenstein G gr-flat module and N an M_{gr}-type module. Then,

(1) *$Tor_i(E, N) = 0$ for each gr-injective right R-module E and for each integer $i \geq 1$.*

(2) *If R is a left GF-gr-closed ring, then there exists an exact sequence $0 \to N \to F \to L \to 0$ in R-gr such that F is an gr-flat and L is an M_{gr}-type module.*

Corollary (4)

Let R be a left GF-gr-closed ring. Let M be a strongly Gorenstein G gr-flat module and N an M_{gr}-type module. Then N is a Gorenstein gr-flat R-module.

Proof of the main theorem

- In view of the inclusions $\mathcal{GF}_{gr}(R) \subseteq \mathcal{GF}_{gr}^{(2)}(R) \subseteq \mathcal{GF}_{i-gr}^{(2)}(R)$, it suffices to prove that $\mathcal{GF}_{i-gr}^{(2)}(R) \subseteq \mathcal{GF}_{gr}(R)$.
- Since R is left *GF*-*gr*-closed, by Theorem 3, $\mathcal{GF}_{gr}(R)$ is stable under direct summands.
- Thus, it suffices, by Proposition 3, to prove that any strongly Gorenstein G *gr*-flat module is Gorenstein *gr*-flat.
- Let M be a strongly Gorenstein G *gr*-flat module. There exists an exact sequence $0 \to M \to G \to M \to 0$ in R-gr such that G is a Gorenstein *gr*-flat module and $Tor_i(E, M) = 0$ for each *gr*-injective right module E and each integer $i \geq 1$ by Proposition 2.
- As G is Gorenstein *gr*-flat, there exists an exact sequence $0 \to G \to F \to G_1 \to 0$ in R-gr such that F is a *gr*-flat module and G_1 is a Gorenstein *gr*-flat module.

Then we get the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & G & \to & M & \to & 0 \\
 & & || & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & F & \to & M_1 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G_1 & = & G_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Hence, M_1 is an M_{gr}-type R-module. It follows from Corollary 4 that M_1 is a Gorenstein gr-flat module. As R is left GF-gr-closed and G_1 is Gorenstein gr-flat, we get M is a Gorenstein gr-flat R-module, as desired.

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