

Natural dualities for finite semigroups

Tim Koussas

Supervisors: Marcel Jackson and Brian Davey (LTU)

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For this to make sense, it needs to be checked that $\mathcal{V}(\mathbf{V}, \mathbf{F})$ is closed under the operations of $\mathbf{F}^{\mathbf{V}}$.

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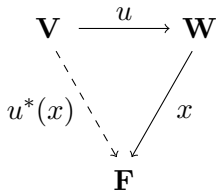
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$$\mathbf{V} \xrightarrow{u} \mathbf{W}$$

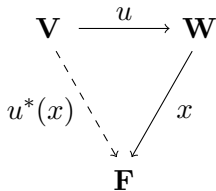
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$$\begin{array}{ccc} \mathbf{V} & \xrightarrow{u} & \mathbf{W} \\ & & \searrow x \\ & & \mathbf{F} \end{array}$$

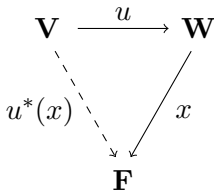
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The map u^* is linear, and moreover $*$: $\mathcal{V} \rightarrow \mathcal{V}$ is a contravariant functor:

$$\text{id}_{\mathbf{V}}^* = \text{id}_{\mathbf{V}^*} \qquad (u \circ v)^* = v^* \circ u^*$$

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This yields a contravariant functor from \mathcal{A} into the category of sets. This functor is often denoted by $\mathcal{A}(-, M)$.

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$$ED(\mathbf{A}) \cong \mathbf{A} \qquad DE(\mathbf{X}) \cong \mathbf{X}$$

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Theorem

For (m)any dually equivalent categories \mathcal{A} and \mathcal{X} , there must exist a natural duality between them.

$$\mathcal{A} \begin{array}{c} \xrightarrow{\mathcal{A}(-, \underline{\mathbf{M}})} \\ \xleftarrow{\mathcal{X}(-, \underline{\widetilde{\mathbf{M}}})} \end{array} \mathcal{X}$$

Given an algebraic category \mathcal{A} , it is sometimes possible to construct a more tractable category \mathcal{X} and a natural duality between \mathcal{A} and \mathcal{X} .

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A basic question: when does an algebra $\underline{\mathbf{M}}$ in \mathcal{A} admit a natural duality between \mathcal{A} and some other category \mathcal{X} ?

$$\begin{array}{ccc}
 & \mathcal{A}(-, \underline{\mathbf{M}}) & \\
 \mathcal{A} & \begin{array}{c} \xrightarrow{\hspace{2cm}} \\ \xleftarrow{\hspace{2cm}} \end{array} & \mathcal{X} \\
 & \mathcal{X}(-, \underline{\mathbf{M}}) &
 \end{array}$$

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It is necessary that $\mathcal{A} \subseteq \mathbb{ISP}(\underline{\mathbf{M}})$, so we usually consider natural dualities between $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$ and some category \mathcal{X} .

$$\begin{array}{ccc}
 & \mathcal{A}(-, \underline{\mathbf{M}}) & \\
 \mathcal{A} & \begin{array}{c} \xrightarrow{\hspace{1.5cm}} \\ \xleftarrow{\hspace{1.5cm}} \end{array} & \mathcal{X} \\
 & \mathcal{X}(-, \underline{\widetilde{\mathbf{M}}}) &
 \end{array}$$

We say that $\underline{\mathbf{M}}$ is **dualisable** if there is a choice of $\underline{\widetilde{\mathbf{M}}}$ giving rise to a duality of the above form, where

$$\mathcal{A} = \text{ISP}(\underline{\mathbf{M}}) \qquad \mathcal{X} = \text{IS}_c\text{P}^+(\underline{\widetilde{\mathbf{M}}})$$

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- This thing

\cdot	e	a	0
e	e	a	0
a	0	0	0
0	0	0	0

Theorem

Let \mathbf{S} be a finite semigroup. Then $\mathbb{HSP}(\mathbf{S})$ is residually large if and only if it contains (up to semigroup duality) a finite semigroup of one of the following types:

- (1) A non-Abelian p -group
- (2) A proper 3-nilpotent semigroup
- (3) A non-orthodox completely simple semigroup
- (4) \mathbf{L}^1
- (5) \mathbf{T}
- (6) \mathbf{R}_3
- (7) $\mathbf{P} \times \mathbf{L}$
- (8) $\mathbf{P} \times \mathbf{G}$, where \mathbf{G} is a non-Abelian group.

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\cdot	0	1	2	3
0	1	1	1	2
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

T

\cdot	a	b	x	y	z	0
a	a	b	x	y	x	0
b	a	b	x	y	y	0
x	0	0	0	0	0	0
y	0	0	0	0	0	0
z	0	0	0	0	0	0
0	0	0	0	0	0	0

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