Natural dualities for finite semigroups

Tim Koussas

Supervisors: Marcel Jackson and Brian Davey (LTU)

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For this to make sense, it needs to be checked that $\boldsymbol{\mathcal{V}}(\mathbf{V}, \mathbf{F})$ is closed under the operations of $\mathbf{F}^V$.

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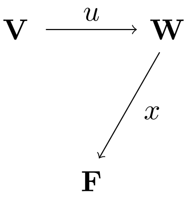

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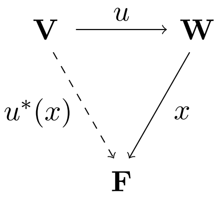

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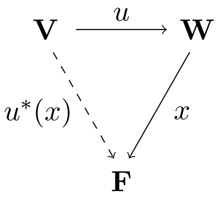

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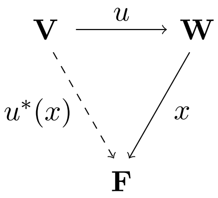

The map u^* is linear, and moreover ${}^*\colon \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}}$ is a contravariant functor:

$$\mathrm{id}_{\mathbf{V}}^* = \mathrm{id}_{\mathbf{V}^*} \qquad\qquad (u \circ v)^* = v^* \circ u^*$$

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This yields a contravariant functor from $\mathcal{A}$ into the category of sets. This functor is often denoted by $\mathcal{A}(-, M)$.

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$$ED(\mathbf{A}) \cong \mathbf{A} \qquad\qquad DE(\mathbf{X}) \cong \mathbf{X}$$

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Theorem

For (m)any dually equivalent categories $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}$, there must exist a natural duality between them.

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Given an algebraic category $\boldsymbol{\mathcal{A}}$, it is sometimes possible to construct a more tractable category $\boldsymbol{\mathfrak{X}}$ and a natural duality between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}$.

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A basic question: when does an algebra $\underline{\mathbf{M}}$ in $\mathcal{A}$ admit a natural duality between $\mathcal{A}$ and some other category $\mathcal{X}$?

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It is necessary that $\boldsymbol{\mathcal{A}} \subseteq \mathbb{ISP}(\underline{\mathbf{M}})$, so we usually consider natural dualities between $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\underline{\mathbf{M}})$ and some category $\boldsymbol{\mathfrak{X}}$.

$$\boldsymbol{\mathcal{A}} \underset{\boldsymbol{\mathfrak{X}}(-,\,\underset{\sim}{\mathbf{M}})}{\overset{\boldsymbol{\mathcal{A}}(-,\,\underline{\mathbf{M}})}{\rightleftarrows}} \boldsymbol{\mathfrak{X}}$$

We say that $\underline{\mathbf{M}}$ is **dualisable** if there is a choice of $\underset{\sim}{\mathbf{M}}$ giving rise to a duality of the above from, where

$$\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\underline{\mathbf{M}}) \qquad\qquad \boldsymbol{\mathfrak{X}} = \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}})$$

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- This thing

$\cdot$	e	a	0
e	e	a	0
a	0	0	0
0	0	0	0

Theorem

Let $\mathbf{S}$ *be a finite semigroup. Then* $\mathbb{HSP}(\mathbf{S})$ *is residually large if and only if it contains (up to semigroup duality) a finite semigroup of one of the following types:*

(1) *A non-Abelian* p*-group*

(2) *A proper 3-nilpotent semigroup*

(3) *A non-orthodox completely simple semigroup*

(4) $\mathbf{L}^1$

(5) $\mathbf{T}$

(6) $\mathbf{R}_3$

(7) $\mathbf{P} \times \mathbf{L}$

(8) $\mathbf{P} \times \mathbf{G}$, *where* $\mathbf{G}$ *is a non-Abelian group.*

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$\cdot$	0	1	2	3
0	1	1	1	2
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

T

$\cdot$	a	b	x	y	z	0
a	a	b	x	y	x	0
b	a	b	x	y	y	0
x	0	0	0	0	0	0
y	0	0	0	0	0	0
z	0	0	0	0	0	0
0	0	0	0	0	0	0

$\mathbf{R}_3$