Graded Steinberg algebras and partial actions

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- McClanahan defined crossed products of C^*- algebras of partial actions of discrete groups in 1995.
- Dokuchaev and Exel introduced partial skew group rings in 2005 which are algebraic analogues of C^*-crossed products by partial actions.

Partial actions

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ON PARTIAL ACTIONS AND GROUPOIDS

FERNANDO ABADIE

(Communicated by David R. Larson)

ABSTRACT. We prove that, as in the case of global actions, any partial action gives rise to a groupoid provided with a Haar system, whose C^*-algebra agrees with the crossed product by the partial action.

INTRODUCTION

The theory of groupoid C^*-algebras has increasingly become a standard field of study in the area of operator algebras. Several large classes of C^*-algebras, such as AF-algebras, Cuntz and Cuntz–Krieger algebras, crossed products of commutative C^*-algebras, etc., may be expressed and studied as C^*-algebras associated with certain groupoids (general references on groupoid C^*-algebras are [16] and [13]). The aim of this paper is to add to the above list all crossed products of commutative C^*-algebras by *partial actions*.

The notion of crossed product by a partial action has its origin in the concept of

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Q : What is the relation between the Steinberg algebra of the groupoid associated to the partial action of a discrete group and partial skew group ring ? (Steinberg algebras of certain groupoids)

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Q : What is the relation between the Steinberg algebra of the groupoid associated to the partial action of a discrete group and partial skew group ring ? (Steinberg algebras of certain groupoids)

- Buss, Exel introduced partial actions of inverse semigroups on algebras in 2014.

A. Buss, R. Exel, Illinois J. Math.

Outline

- partial actions and partial skew inverse semigroup rings

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- interplay

Inverse semigroups

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$\leq$: partial order relation on $\mathcal{S}$ s.t. for $s, t \in \mathcal{S}$,

$$s \leq t \Longleftrightarrow s = ts^*s \Longleftrightarrow s = ss^*t$$

.

Partial actions

Definition

A partial action of $\mathcal{S}$ on a set X is $\pi = (\pi_s, X_s, X)_{s\in\mathcal{S}}$ with $X_s \subseteq X$ a subset and $\pi_s : X_{s^*} \to X_s$ a bijection s.t. for all $s, t \in \mathcal{S}$

(i) $\pi_s^{-1} = \pi_{s^*}$;

(ii) $\pi_s(X_{s^*} \cap X_t) \subseteq X_{st}$;

(iii) if $s \leq t$, then $X_s \subseteq X_t$;

(iv) For every $x \in X_{t^*} \cap X_{t^*s^*}$, $\pi_s(\pi_t(x)) = \pi_{st}(x)$.

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For topological setting, X_s is an open set and π_s a homeomorphism of topological spaces. If X is an algebra/ring, then X_s an ideal and π_s an isomorphism of algebras.

A. Buss, R. Exel, Illinois J. Math.

Partial skew inverse semigroup rings

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Define $\mathcal{L}$ as the set of all formal forms

$$\sum_{s\in\mathcal{S}} a_s\delta_s,$$

where $a_s \in A_s$ and δ_s are symbols.
addition: $+$
multiplication: the linear extension of

$$(a_s\delta_s)(a_t\delta_t) = \pi_s\big(\pi_{s^{-1}}(a_s)a_t\big)\delta_{st}.$$

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If each ideal A_s is idempotent, then $\mathcal{L}$ is associative.

R. Exel, F. Vieira, J. Math. Anal. Appl.

Partial skew inverse semigroup rings

Definition

Let $\pi = (\pi_s, A_s, A)_{s \in \mathcal{S}}$ be an action of an inverse semigroup $\mathcal{S}$ on an algebra A. Consider $\mathcal{N} = \langle a\delta_s - a\delta_t : a \in A_s, s \leq t \rangle$, which is the ideal generated by $a\delta_s - a\delta_t$. The partial skew inverse semigroup ring $A \rtimes_\pi \mathcal{S}$ is defined as the quotient ring $\mathcal{L}/\mathcal{N}$.

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Grading

G: a group

$w : \mathcal{S} \setminus \{0\} \to G$: a function s.t.

$$w(st) = w(s)w(t)$$

for $s, t \in \mathcal{S}$ with $st \neq 0$. If $\mathcal{S}$ does not have a zero element then the function w is a homomorphism from $\mathcal{S}$ to G.

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$A \rtimes_\pi \mathcal{S}$ is G-graded: $a_s\delta_s \in \mathcal{L}$ with $a_s \in A_s$ are homogeneous elements of degree $w(s)$. For $s \leq t$,
$w(s) = w(t)w(s^*)w(s) = w(t)$.

Partial skew group rings

X: a Hausdorff topological space

R: a unital commutative ring with a discrete topology

$C_R(X)$: the set of R-valued continuous function (i.e., locally constant) with compact support (idempotent)

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A partial action $(\pi_g, X_g, X)_{g \in G}$ of G on X induces a partial action $(\pi_g, C_R(X_g), C_R(X))_{g \in G}$ of G on an algebra $C_R(X)$ s.t. $\pi_g : C_R(X_{g^{-1}}) \to C_R(X_g)$ is given by $\pi_g(f) = f \circ \pi_g^{-1}$.

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$C_R(X) \rtimes_\pi G$ is called a partial skew group ring.

An ample groupoid

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- An open bisection of $\mathcal{G}$ is an open subset $U \subseteq \mathcal{G}$ s.t. $d|_U$ and $r|_U$ are homoemorphisms onto an open subset of $\mathcal{G}^{(0)}$.
- $\mathcal{G}$ is an ample groupoid, if $\exists$ a basis consisting of compact open bisections for its topology and the domain map d is a local homeomorphism.

Steinberg algebras

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The Steinberg algebra is the R-algebra

$$A_R(\mathcal{G}) = \text{span}\{1_U : U \text{ is a compact open bisection of } \mathcal{G}\},$$

where 1_U denotes the characteristic function on U; addition is pointwise and multiplication is given by convolution

$$(f * g)(\gamma) = \sum_{\{\alpha\beta=\gamma\}} f(\alpha)g(\beta).$$

Steinberg, Adv. Math.

Clark, Farthing, Sims, Tomforde, Semigroup Forum

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Γ: a discrete group
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$$A_R(\mathcal{G})_\gamma = \{f \in A_R(\mathcal{G}) \mid f(x) \neq 0 \Rightarrow x \in c^{-1}(\gamma)\}.$$

L.O. Clark, A. Sims, J. Pure Appl. Algebra

The interplay

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$\mathcal{G}^{(0)}$: the unit space of $\mathcal{G}$

$U \subseteq \mathcal{G}^{(0)}$: an open invariant subset (invariant: for any $\gamma \in \mathcal{G}$, $d(\gamma) \in U \Leftrightarrow r(\gamma) \in U$.)

Set

$$\mathcal{G}^{(h)} = \{B \mid B \text{ is a graded compact open bisection of } \mathcal{G}\}. \quad (0.1)$$

Then $\mathcal{G}^{(h)}$ is an inverse semigroup s.t. $B^* = B^{-1} = \{b^{-1} | b \in B\}$, $B \in \mathcal{G}^{(h)}$.

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$\Longrightarrow \pi = (\pi_B, U_B, U)_{B \in \mathcal{G}^{(h)}}$ is a partial action of $\mathcal{G}^{(h)}$ on U.

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π induces $(\pi_B, C_R(U_B), C_R(U))_{B \in \mathcal{G}^{(h)}}$ of $\mathcal{G}^{(h)}$ on an algebra $C_R(U)$, where $\pi_B : C_R(U_{B^{-1}}) \to C_R(U_B)$ is given by $\pi_B(f) = f \circ \pi_B^{-1}$.

Result

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Theorem

Let $\mathcal{G}$ be a Γ-graded ample Hausdorff groupoid, $U \subseteq \mathcal{G}^{(0)}$ an open invariant subset and $\pi = (\pi_B, C_R(U_B), C_R(U))_{B\in\mathcal{G}^{(h)}}$ the induced partial action of $\mathcal{G}^{(h)}$ on $C_R(U)$. Then

$$A_R(\mathcal{G}_U) \cong_{\mathrm{gr}} C_R(U) \rtimes_\pi \mathcal{G}^{(h)}.$$

In particular, $A_R(\mathcal{G}) \cong_{\mathrm{gr}} C_R(\mathcal{G}^{(0)}) \rtimes_\pi \mathcal{G}^{(h)}$.

Consequence

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G-graded groupoid

$$\mathcal{G}_X = \bigcup_{g \in G} g \times X_g,$$

s.t. $(g, x)(h, y) = (gh, x)$ if $y = \pi_{g^{-1}}(x)$,
$(g, x)^{-1} = (g^{-1}, \pi_{g^{-1}}(x))$.

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If X_g's are clopen subsets of X, then $A_R(\mathcal{G}_X) \cong_{\mathrm{gr}} C_R(X) \rtimes_\pi G$.

Thank you!

Thank you very much for your attention !