

Permutations sorted by a finite and an infinite stack in series

Yoong Kuan Goh(Andrew)
Supervisor: Associate Professor Murray Elder

University of Technology Sydney

27 November 2017

Definitions

Permutation

In this work, a *permutation* is a bijection from an ordered set to itself.

Definitions

Permutation

In this work, a *permutation* is a bijection from an ordered set to itself.

Eg: take $\{1, \dots, n\}$ with $<$. We write a permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ as 312.

Definitions

Permutation

In this work, a *permutation* is a bijection from an ordered set to itself.

Eg: take $\{1, \dots, n\}$ with $<$. We write a permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ as 312.

Order isomorphic

Two permutations with same relative ordering are *order isomorphic*.

Eg: 312 and 825 are order isomorphic as they both have relative ordering *BIG, small, Middle*.

Definitions

Permutation

In this work, a *permutation* is a bijection from an ordered set to itself.

Eg: take $\{1, \dots, n\}$ with $<$. We write a permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ as 312.

Order isomorphic

Two permutations with same relative ordering are *order isomorphic*.

Eg: 312 and 825 are order isomorphic as they both have relative ordering *BIG, small, Middle*.

Pattern avoiding

A permutation $p = p_1 \dots p_n$ *contains* q if it has a sub-permutation $p_{i_1} \dots p_{i_s}$ that is order isomorphic to q . If not, we say p *avoids* q .

Definitions

Permutation

In this work, a *permutation* is a bijection from an ordered set to itself.

Eg: take $\{1, \dots, n\}$ with $<$. We write a permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ as 312.

Order isomorphic

Two permutations with same relative ordering are *order isomorphic*.

Eg: 312 and 825 are order isomorphic as they both have relative ordering *BIG, small, Middle*.

Pattern avoiding

A permutation $p = p_1 \dots p_n$ *contains* q if it has a sub-permutation $p_{i_1} \dots p_{i_s}$ that is order isomorphic to q . If not, we say p *avoids* q .

Eg: 1234 avoids 21.

Avoidance set

Let S_n be the set of all permutations of $\{1, \dots, n\}$ and $S_\infty = \bigcup_{n \geq 1} S_n$. If $B \subseteq S_\infty$, $Av(B)$ denotes the set of all permutations in S_∞ that avoid every permutation in B .

Avoidance set

Let S_n be the set of all permutations of $\{1, \dots, n\}$ and $S_\infty = \bigcup_{n \geq 1} S_n$. If $B \subseteq S_\infty$, $Av(B)$ denotes the set of all permutations in S_∞ that avoid every permutation in B .

Eg: $Av(\{12, 123\})$ is the set of all decreasing permutations.

Avoidance set

Let S_n be the set of all permutations of $\{1, \dots, n\}$ and $S_\infty = \bigcup_{n \geq 1} S_n$. If $B \subseteq S_\infty$, $Av(B)$ denotes the set of all permutations in S_∞ that avoid every permutation in B .

Eg: $Av(\{12, 123\})$ is the set of all decreasing permutations.

Pattern-avoidance class

A set S of permutations is a *pattern avoidance class* if there is some set B of permutations such that $S = Av(B)$.

Avoidance set

Let S_n be the set of all permutations of $\{1, \dots, n\}$ and $S_\infty = \bigcup_{n \geq 1} S_n$. If $B \subseteq S_\infty$, $Av(B)$ denotes the set of all permutations in S_∞ that avoid every permutation in B .

Eg: $Av(\{12, 123\})$ is the set of all decreasing permutations.

Pattern-avoidance class

A set S of permutations is a *pattern avoidance class* if there is some set B of permutations such that $S = Av(B)$.

Antichain

B is an *antichain* if for all $q_i, q_j \in B$, q_i avoids q_j .

Avoidance set

Let S_n be the set of all permutations of $\{1, \dots, n\}$ and $S_\infty = \bigcup_{n \geq 1} S_n$. If $B \subseteq S_\infty$, $Av(B)$ denotes the set of all permutations in S_∞ that avoid every permutation in B .

Eg: $Av(\{12, 123\})$ is the set of all decreasing permutations.

Pattern-avoidance class

A set S of permutations is a *pattern avoidance class* if there is some set B of permutations such that $S = Av(B)$.

Antichain

B is an *antichain* if for all $q_i, q_j \in B$, q_i avoids q_j .

Basis

B is called a *basis* for S if $S = Av(B)$ and B is an antichain.

Avoidance set

Let S_n be the set of all permutations of $\{1, \dots, n\}$ and $S_\infty = \bigcup_{n \geq 1} S_n$. If $B \subseteq S_\infty$, $Av(B)$ denotes the set of all permutations in S_∞ that avoid every permutation in B .

Eg: $Av(\{12, 123\})$ is the set of all decreasing permutations.

Pattern-avoidance class

A set S of permutations is a *pattern avoidance class* if there is some set B of permutations such that $S = Av(B)$.

Antichain

B is an *antichain* if for all $q_i, q_j \in B$, q_i avoids q_j .

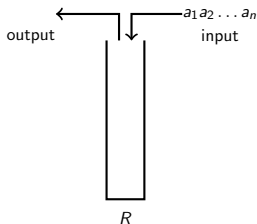
Basis

B is called a *basis* for S if $S = Av(B)$ and B is an antichain.

Eg: the pattern avoidance class with basis $\{12, 21\}$ is ...

Stack sorting

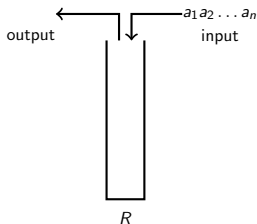
Knuth [3]: a permutation can be *sorted* by passing it through an infinite stack



if and only if it avoids

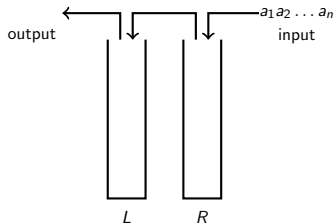
Stack sorting

Knuth [3]: a permutation can be *sorted* by passing it through an infinite stack

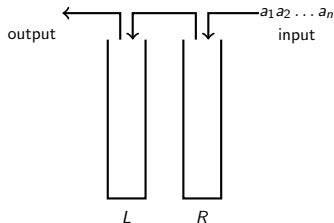


if and only if it avoids 231.

Murphy [4] extended Knuth's research by using two infinite stacks in series and proved that the basis for sorted permutations is

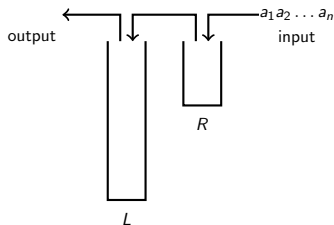


Murphy [4] extended Knuth's research by using two infinite stacks in series and proved that the basis for sorted permutations is

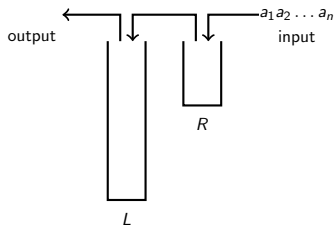


infinite.

Elder [1] considered the (easier) problem of one finite and one infinite stack in series, and proved that for $(2, \infty)$ the basis is



Elder [1] considered the (easier) problem of one finite and one infinite stack in series, and proved that for $(2, \infty)$ the basis is



finite.

$S(2, \infty)$ basis

23451, 23541, 24531, 32451, 32541, 42531,
245163, 246153, 425163, 426153, 456231, 546231,
2531674, 2531764, 2671453, 5231674, 5231764, 6271453,
27318564, 72318564

He also proved that if σ is in the basis for $S(k, \infty)$ and has length n then either σ or $(213)_n\sigma$ is in the basis for $S(k + 1, \infty)$,
so if depth k is infinitely based then so is $k + 1$.

He also proved that if σ is in the basis for $S(k, \infty)$ and has length n then either σ or $(213)_n\sigma$ is in the basis for $S(k + 1, \infty)$,

so if depth k is infinitely based then so is $k + 1$.

(Here the notation $(213)_n$ means add n to each entry.)

He also proved that if σ is in the basis for $S(k, \infty)$ and has length n then either σ or $(213)_n\sigma$ is in the basis for $S(k + 1, \infty)$,

so if depth k is infinitely based then so is $k + 1$.

(Here the notation $(213)_n$ means add n to each entry.)

Open research question

What is the breakpoint between a stack of depth 2 to infinite depth for the basis to change from finite to infinite?

Computational Approach for $S(3, \infty)$

I wrote a computer program for a stack of depth 3 and an infinite stack in series. The code works by enumerating words in the three stack moves, to find the basis for small lengths.

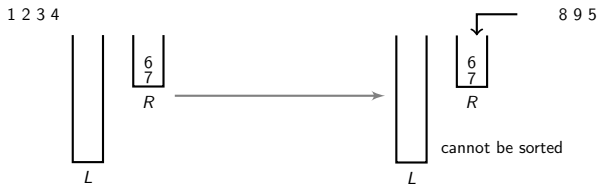
Computational Approach for $S(3, \infty)$

I wrote a computer program for a stack of depth 3 and an infinite stack in series. The code works by enumerating words in the three stack moves, to find the basis for small lengths.

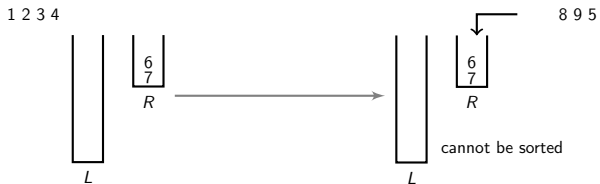
Perm length	# sortable perms	# Size of basis
5	120	0
6	711	9
7	4700	83
8	33039	169
9	239800	345
10	1769019	638
11	13160748	1069
12	98371244	1980
13	737463276	3901

The pattern $(2\ 4\ 3\ 7\ 6\ 1)(8\ 9\ 5)$ is in the basis for $S(3, \infty)$:

The pattern $(2\ 4\ 3\ 7\ 6\ 1)\ (8\ 9\ 5)$ is in the basis for $S(3, \infty)$:



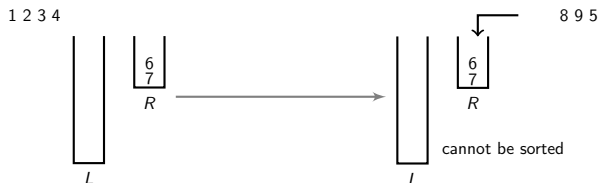
The pattern $(2\ 4\ 3\ 7\ 6\ 1)\ (8\ 9\ 5)$ is in the basis for $S(3, \infty)$:



Question

How to extend this to an infinite antichain?

The pattern $(2\ 4\ 3\ 7\ 6\ 1)\ (8\ 9\ 5)$ is in the basis for $S(3, \infty)$:



Question

How to extend this to an infinite antichain?

$$G_0 = (2\ 4\ 3\ 7\ 6\ 1)\ (10\ 5\ 9)\ (13\ 12\ 8)\ (14\ 15\ 11)$$

A family of permutations $\mathcal{G} = \{G_i \mid i \in \mathbb{N}\}$ for $S(3, \infty)$

Define

$$P = 2\ 4\ 3\ 7\ 6\ 1$$

$$x_j = (10\ 5\ 9)_{6j}$$

$$y_j = (13\ 12\ 8)_{6j}$$

$$S_i = (14\ 15\ 11)_{6i}$$

$$G_i = P\ x_0y_0\ x_1y_1 \dots x_iy_i\ S_i$$

A family of permutations $\mathcal{G} = \{G_i \mid i \in \mathbb{N}\}$ for $S(3, \infty)$

Define

$$P = 2\ 4\ 3\ 7\ 6\ 1$$

$$x_j = (10\ 5\ 9)_{6j}$$

$$y_j = (13\ 12\ 8)_{6j}$$

$$S_i = (14\ 15\ 11)_{6i}$$

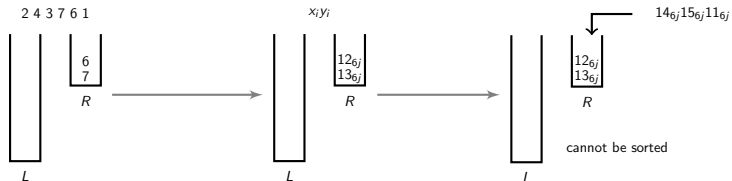
$$G_i = P\ x_0y_0\ x_1y_1 \dots x_iy_i\ S_i$$

The first three terms are

$$G_0 = 2\ 4\ 3\ 7\ 6\ 1\ (10\ 5\ 9)\ (13\ 12\ 8)\ 14\ 15\ 11$$

$$G_1 = 2\ 4\ 3\ 7\ 6\ 1\ (10\ 5\ 9)\ (13\ 12\ 8)\ (16\ 11\ 15)\ (19\ 18\ 14)\ 20\ 21\ 17$$

$$G_2 = 2\ 4\ 3\ 7\ 6\ 1\ (10\ 5\ 9)\ (13\ 12\ 8)\ (16\ 11\ 15)\ (19\ 18\ 14)\ (22\ 17\ 21) \\ (25\ 24\ 20)\ 26\ 27\ 23$$



Proposition

$G_i \notin S(3, \infty)$ for all $i \in \mathbb{N}$

Proposition

Let G'_i be a permutation obtained by removing a single entry from G_i .
Then $G'_i \in S(3, \infty)$.

In order to prove this, we give a deterministic sorting procedure in each case:

- 1 Term removed from P .
- 2 Term removed from $x_S y_S$.
- 3 Term removed from S_i .

The detail proof is in my paper in arXiv [identifier: 1711.06040] [2]

Theorem

The family of permutations $\mathcal{G} = \{G_i \mid i \in \mathbb{N}\}$ is an infinite basis in $S(3, \infty)$.

Finite to Infinite Bases

Recall:

Lemma

If $\sigma \in \mathcal{B}_t$ has length n then one of $\sigma, (213)_n\sigma$ is in \mathcal{B}_{t+1} .

Theorem

The set of permutations that can be sorted using a stack of depth $t \in \mathbb{N}_+$ and an infinite stack in series is finitely based if and only if $t \in \{1, 2\}$.

References I



M. Elder.

Permutations generated by a stack of depth 2 and an infinite stack in series.
E-JC, 9(R68):112, 2006.



M. Elder and Y. K. Goh.

Permutations sorted by a finite and an infinite stack in series.
ArXiv e-prints, Nov. 2017.



D. E. Knuth.

The Art of Computer Programming, Volume 3: (2Nd Ed.) Sorting and Searching.
Addison Wesley Longman Publishing Co., Inc., Redwood City, CA, USA, 1998.



M. Murphy.

Restricted permutations, antichains, atomic classes, stack sorting.
PhD thesis, University of St Andrew, 2002.