Permutations sorted by a finite and an infinite stack in series

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Eg: 1234 avoids 21.

Avoidance set

Let S_n be the set of all permutations of $\{1, \ldots, n\}$ and $S_\infty = \bigcup_{n \geq 1} S_n$. If $B \subseteq S_\infty$, $Av(B)$ denotes the set of all permutations in S_∞ that avoid every permutation in B.

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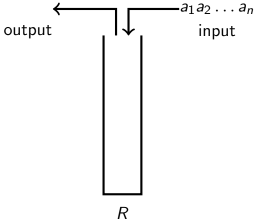
Eg: the pattern avoidance class with basis $\{12, 21\}$ is ...

Stack sorting

Knuth [3]: a permutation can be *sorted* by passing it through an infinite stack



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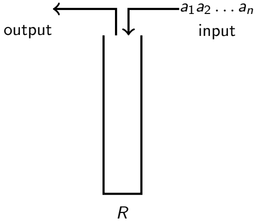
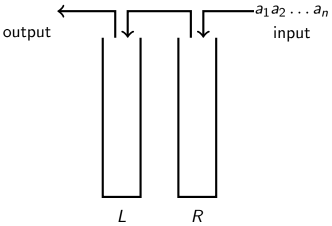


if and only if it avoids 231.

Murphy [4] extended Knuth's research by using two infinite stacks in series and proved that the basis for sorted permutations is



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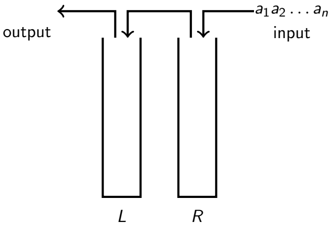

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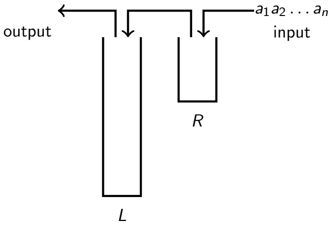
Elder [1] considered the (easier) problem of one finite and one infinite stack in series, and proved that for $(2, \infty)$ the basis is



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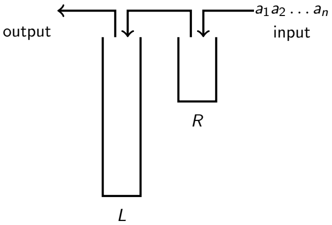

finite.

$S(2, \infty)$ basis

23451, 23541, 24531, 32451, 32541, 42531,
245163, 246153, 425163, 426153, 456231, 546231,
2531674, 2531764, 2671453, 5231674, 5231764, 6271453,
27318564, 72318564

He also proved that if σ is in the basis for $S(k, \infty)$ and has length n then either σ or $(213)_n\sigma$ is in the basis for $S(k+1, \infty)$,

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Open research question

What is the breakpoint between a stack of depth 2 to infinite depth for the basis to change from finite to infinite?

Computational Approach for $S(3, \infty)$

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Perm length	# sortable perms	# Size of basis
5	120	0
6	711	9
7	4700	83
8	33039	169
9	239800	345
10	1769019	638
11	13160748	1069
12	98371244	1980
13	737463276	3901

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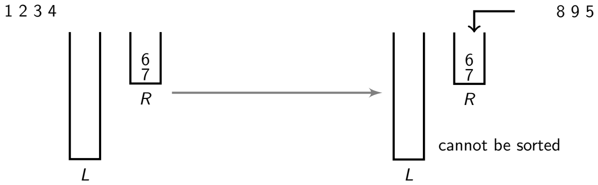

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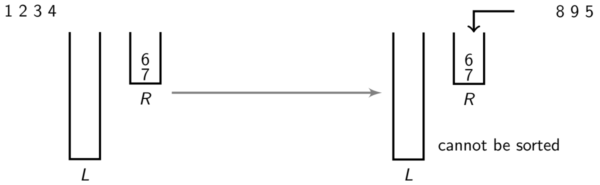

Question

How to extend this to an infinite antichain?

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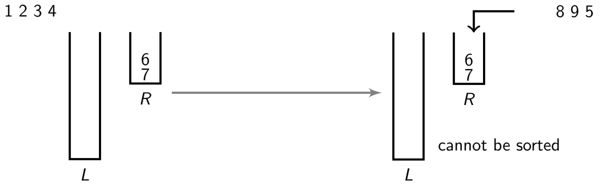

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How to extend this to an infinite antichain?

$G_0 = (2\ 4\ 3\ 7\ 6\ 1)\ (10\ 5\ 9)\ (13\ 12\ 8)\ (14\ 15\ 11)$

A family of permutations $\mathscr{G} = \{G_i \mid i \in \mathbb{N}\})$ for $S(3, \infty)$

Define

$$\begin{aligned} P &= 2\ 4\ 3\ 7\ 6\ 1 \\ x_j &= (10\ 5\ 9)_{6j} \\ y_j &= (13\ 12\ 8)_{6j} \\ S_i &= (14\ 15\ 11)_{6i} \\ \\ G_i &= P\ x_0 y_0\ x_1 y_1 \ldots x_i y_i\ S_i \end{aligned}$$

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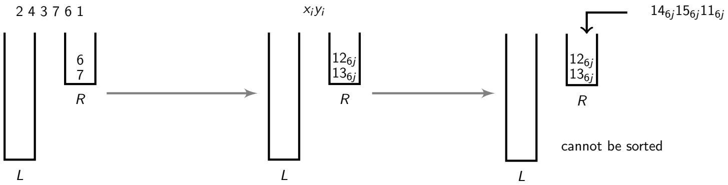
The first three terms are

$$\begin{aligned} G_0 &= 2\ 4\ 3\ 7\ 6\ 1\ (10\ 5\ 9)\ (13\ 12\ 8)\ 14\ 15\ 11 \\ G_1 &= 2\ 4\ 3\ 7\ 6\ 1\ (10\ 5\ 9)\ (13\ 12\ 8)\ (16\ 11\ 15)\ (19\ 18\ 14)\ 20\ 21\ 17 \\ G_2 &= 2\ 4\ 3\ 7\ 6\ 1\ (10\ 5\ 9)\ (13\ 12\ 8)\ (16\ 11\ 15)\ (19\ 18\ 14)\ (22\ 17\ 21) \\ &\quad (25\ 24\ 20)\ 26\ 27\ 23 \end{aligned}$$



Proposition

$G_i \notin S(3, \infty)$ for all $i \in \mathbb{N}$

Proposition

Let G_i' be a permutation obtained by removing a single entry from G_i. Then $G_i' \in S(3, \infty)$.

In order to prove this, we give a deterministic sorting procedure in each case:

1. Term removed from P.
2. Term removed from $x_s y_s$.
3. Term removed from S_i.

The detail proof is in my paper in arXiv [identifier: 1711.06040] [2]

Theorem

The family of permutations $\mathscr{G} = \{G_i \mid i \in \mathbb{N}\}$ is an infinite basis in $S(3, \infty)$.

Finite to Infinite Bases

Recall:

Lemma

If $\sigma \in \mathscr{B}_t$ has length n then one of $\sigma, (213)_n\sigma$ is in $\mathscr{B}_{t+1}$.

Theorem

The set of permutations that can be sorted using a stack of depth $t \in \mathbb{N}_+$ and an infinite stack in series is finitely based if and only if $t \in \{1, 2\}$.

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