Parabolic subgroups of Artin–Tits groups of spherical type

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Outline

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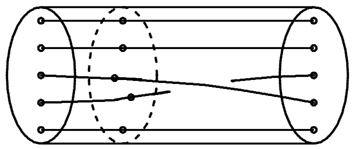
Artin–Tits groups of spherical type

- An Artin–Tits group is a group defined by a presentation

$$A_S = \left\langle S \;\middle|\; \underbrace{sts\ldots}_{m_{s,t}\text{ factors}} = \underbrace{tst\ldots}_{m_{s,t}\text{ factors}} \quad (s,t \in S) \right\rangle$$

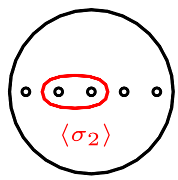
  where S is a finite set and $M = \left(m_{s,t}\right)_{s,t}$ is a Coxeter matrix.
- Adding the relations $s^2 = 1$ for all $s \in S$, one obtains the associated Coxeter group W_S.
- A_S is called of spherical type if W_S is finite...
- ...and irreducible if $A_S = G \times H$ implies $G = 1$ or $H = 1$.
- The most famous examples are the Artin braid groups B_n:
  - $S = \{\sigma_1, \ldots, \sigma_{n-1}\}$
  - $m_{\sigma_i,\sigma_j} = \begin{cases} 2 & \text{, if } |i-j| \neq 1 \\ 3 & \text{, if } |i-j| = 1 \end{cases}$



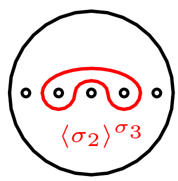
σ_2

Parabolic subgroups

- A standard parabolic subgroup of A_S is a subgroup generated by a subset $T \subseteq S$. It can be identified with the group A_T.
- A parabolic subgroup of A_S is a subgroup that is conjugate to a standard parabolic subgroup.
- <u>For braid groups</u>, parabolic subgroups are easily described:
  - An irreducible standard parabolic subgroup A_T is generated by a set T of generators acting on adjacent punctures. So it can be characterised by a round curve enclosing these punctures.



  - Conjugating A_T merely distorts this curve. So an irreducible parabolic subgroup corresponds to a non-degenerate simple closed curve.



- The above description in terms of curves relies on the geometric interpretation of braids.

Braid groups: curves (= irreducible parabolics) are useful

Curve complex $\mathcal{C}$

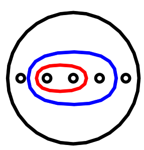
- points: isotopy classes of non-degenerate simple closed curves
- d-simplex: $d+1$ points with pairwise disjoint representatives
- $\mathcal{C}$ is hyperbolic. (Masur–Minsky, 1999)
- Action of B_n on $\mathcal{C}$ allows geometric arguments to study B_n: Nielsen–Thurston classification, structure of centralisers, ...

Canonical reduction system CRS_α for $\alpha \in B_n$

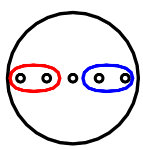
- Let $\mathcal{C}_\alpha$ be the set of points of $\mathcal{C}$ preserved by the action of α.
- Call $\gamma \in \mathcal{C}_\alpha$ essential, if γ and every other $\gamma' \in \mathcal{C}_\alpha$ have disjoint representatives. $CRS_\alpha := \{\gamma \in \mathcal{C}_\alpha \mid \gamma \text{ is essential}\}$.
- If α is *non-periodic* and $\mathcal{C}_\alpha$ is non-empty, then CRS_α is non-empty. (Birman–Lubotzky–McCarthy, 1983)
- This allows to decompose α into canonical simpler components and use induction arguments.

What does "curves being disjoint" mean?

- The basic idea is obvious: Use irreducible parabolic subgroups as an algebraic equivalent of simple closed curves.
- In the braid group, there are two ways in which two parabolic subgroups P and Q can correspond to disjoint curves:



$P \subseteq Q$ or $Q \subseteq P$



$P \cap Q = \{1\}$
and $pq = qp$ for $p \in P$, $q \in Q$

- This looks a bit messy...

Central elements characterising parabolic subgroups

Definition (central element associated to a parabolic subgroup)

If P is an irreducible parabolic subgroup, let z_P be the generator of its centre that has positive exponent sum. If $P = P_1 \times \cdots \times P_k$ for irreducible parabolic subgroups $P_1, \ldots, P_k$, let $z_P = z_{P_1} \cdots z_{P_k}$.

- These central elements characterise the parabolic subgroups:

Lemma (Godelle 2003, Cumplido 2017)

For parabolic subgroups P and Q:

$$P^x = Q \quad \text{if and only if} \quad (z_P)^x = z_Q.$$

- For braids: The central element of the parabolic subgroup associated to a curve is the Dehn twist along the curve. Curves are disjoint if and only if the central elements of their associated parabolic subgroups commute.

Complex of irreducible parabolic subgroups

Definition

The complex of irreducible parabolic subgroups is the simplicial complex in which a d-simplex is a set $\{P_0, \ldots, P_d\}$ of irreducible parabolic subgroups such that z_{P_i} commutes with z_{P_j} for every $0 \leq i, j \leq d$.

- For braid groups, the complex of irreducible parabolic subgroups is isomorphic to the complex of curves. In particular, it is hyperbolic.
- We suspect that the complex of irreducible parabolic subgroups is hyperbolic in general.
- Proving this would allow to generalise many results for braid groups to all Artin–Tits groups of spherical type.

Minimal parabolic subgroup containing an element

- Let A_S be an Artin–Tits group of spherical type and $\alpha \in A_S$.

Theorem (minimal parabolic subgroup containing an element)

There is a unique parabolic subgroup P_α that is minimal (by inclusion) among all parabolic subgroups containing α.

- These are invariant under taking powers and roots of α:

Theorem (minimal parabolic subgroup of a power / root)

If $m \in \mathbb{Z} \setminus \{0\}$, then $P_{\alpha^m} = P_\alpha$.

- Which in turn has an interesting consequence:

Corollary (parabolic subgroups are closed under taking roots)

If α belongs to a parabolic subgroup P, and $\beta \in A_S$ is such that $\beta^m = \alpha$ for some $m \in \mathbb{Z} \setminus \{0\}$, then $\beta \in P$.

Poset of parabolic subgroups

- Existence and uniqueness of P_α make it possible to get a good description of the poset of parabolic subgroups with respect to inclusion:

Theorem (intersection of parabolic subgroups)

If P and Q are two parabolic subgroups of A_S, then $P \cap Q$ is also a parabolic subgroup of A_S.

Theorem (lattice of parabolic subgroups)

The poset of parabolic subgroups of A_S with respect to inclusion is a lattice. (That is, least common upper bounds and greatest common lower bounds exist and are unique.)

Garside structures

Warning: I skip details / cut a few (small) corners in this section.

- We say that x is a divisor of y, if $x^{-1}y$ can be written as a positive element in the generators.
- An element of a Garside group can be described by a canonical sequence of divisors of a distinguished Garside element.
- For an Artin–Tits group, the longest element of the Coxeter group determines one possible Garside element; call this Δ. With Δ as Garside element, elements of the Artin–Tits group are described by sequences of elements of the Coxeter group.
- Other Garside elements are given by the positive powers of Δ.
- The np-normal form of α with respect to a Garside element is a canonical representation of the form $\alpha = x_s^{-1} \cdots x_1^{-1} y_1 \cdots y_t$ where the x_i and y_j are divisors of the Garside element, subject to certain coprimality conditions.

Twisted cycling and decycling

Definition

If $\alpha = x_s^{-1} \cdots x_1^{-1} y_1 \cdots y_t$ is the np-normal form of $\alpha \in A_S$ with respect to the Garside element Δ^m, we call

- $\widetilde{c}_{\Delta^m}(\alpha) := x_{s-1}^{-1} \cdots x_1^{-1}\, y_1 \cdots y_t\, x_s^{-1}$ the twisted cycling of α
- $d_{\Delta^m}(\alpha) := y_t\, x_s^{-1} \cdots x_1^{-1}\, y_1 \cdots y_{t-1}$ the decycling of α

with respect to Δ^m.

- Twisted cycling and decycling are conjugation by x_s^{-1} and y_t^{-1}, that is, conjugation by the inverse of a positive element.

Conjugacy invariants

- An element α in general has infinitely many conjugates, but there are some distinguished finite subsets of conjugates:
  - The set $C^+(\alpha)$ of positive conjugates. (This may be empty.)
  - The set $SSS_{\Delta^m}(\alpha)$ of conjugates for which the length of the np-normal form for the Garside element Δ^m is minimal.
  - The set $RSSS_{\Delta^m}(\alpha)$ of conjugates that are in a period under twisted cycling and decycling for the Garside element Δ^m.
- One always has $RSSS_{\Delta^m}(\alpha) \subseteq SSS_{\Delta^m}(\alpha)$.
- Each of these invariants can be described by a directed graph:
  - The vertices are the conjugates contained in the invariant.
  - The edges are given by minimal positive conjugating elements.
- Starting from any element, these invariants can be reached by finitely many twisted cyclings and decyclings.
- If $\alpha, \alpha^x, \alpha^y$ are contained in one of these invariants, then so is $\alpha^{x \wedge y}$. (This is sort of a "convexity" property.)

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Tool for understanding conjugacy invariants: transport

- Twisted cycling in $RSSS_{\Delta^m}(\alpha)$ is actually a functor: If $\beta, \beta^x \in RSSS_{\Delta^m}(\alpha)$, there is a unique element $x^{(1)}$ with $\widetilde{c}_{\Delta^m}(\beta)^{x^{(1)}} = \widetilde{c}_{\Delta^m}(\beta^x)$ ("transport"). Define $x^{(N)}$ inductively.

Lemma (transport is periodic; G. 2005)

There is $N \in \mathbb{N}^+$ with $\widetilde{c}^{\,N}_{\Delta^m}(\beta) = \beta$, $\widetilde{c}^{\,N}_{\Delta^m}(\beta^x) = \beta^x$ and $x^{(N)} = x$.

Definition (conjugating elements along cycles)

For $N \in \mathbb{N}^+$, denote by $\widetilde{C}^{(N)}_{\Delta^m}(\beta)$ the product of the N consecutive conjugating elements for twisted cycling, starting at the element β.

Corollary (conjugation of conjugating elements along cycles)

With N as in the above Lemma, $\left(\widetilde{C}^{(N)}_{\Delta^m}(\beta)\right)^x = \widetilde{C}^{(N)}_{\Delta^m}(\beta^x)$.

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Support of an element

- Recall $A_S = \left\langle S \;\middle|\; \underbrace{sts\ldots}_{m_{s,t}\text{ factors}} = \underbrace{tst\ldots}_{m_{s,t}\text{ factors}} \quad (s,t\in S)\right\rangle$
- The monoid A_S^+ defined by this presentation embeds into A_S, and both sides of each relation involve the same generators.

  So the following definition makes sense:

Definition (support of an element)

- For $x \in A_S^+$, define $\mathrm{supp}(x)$ as the set of generators that appear in some (hence any) any positive word representing x.
- If $\alpha = x_s^{-1}\cdots x_1^{-1}y_1\cdots y_t$ is the np-normal form of α, define $\mathrm{supp}(\alpha) := \mathrm{supp}(x_1\cdots x_s)\cup \mathrm{supp}(y_1\cdots y_t)$.

Power Garside structures

- We want the np-normal form of any element of interest to be of the form $x^{-1}y$, where x and y divide the Garside element.
- This means working with all Garside structures for the Garside elements $\Delta, \Delta^2, \Delta^3, \ldots$ simultaneously.

Proposition (simultaneous conjugacy invariants)

- $SSS_\infty(\alpha) := \bigcap_{m\geq 1} SSS_{\Delta^m}(\alpha)$ *is non-empty.*
- $RSSS_\infty(\alpha) := \bigcap_{m\geq 1} RSSS_{\Delta^m}(\alpha)$ *is non-empty.*

Main proof ingredients:

- explicit structure of the conjugating elements for $\widetilde{c}_{\Delta^m}$ and d_{Δ^m}
- convexity property

Positive conjugates in a proper parabolic subgroup

Lemma

If α is contained in a proper standard parabolic subgroup, then so is each of $\widetilde{c}_{\Delta^m}(\alpha)$ and $d_{\Delta^m}(\alpha)$.

Main proof ingredients:

- explicit structure of the conjugating elements for $\widetilde{c}_{\Delta^m}$ and d_{Δ^m}

Corollary (parabolics containing positive conjugates are standard)

If α is contained in a proper parabolic subgroup, then each positive conjugate of α belongs to a proper standard parabolic subgroup.
If v and w are positive conjugates of α, then $v^x = w$ implies $(\mathrm{supp}(v))^x = \mathrm{supp}(w)$.

Main proof ingredients:

- careful analysis of labels of arrows in positive conjugate graph

Minimal parabolic subgroup containing a given element α

Definition (get parabolic from the support of a simplest conjugate)

Pick $\alpha' = \beta^{-1}\alpha\beta \in RSSS_\infty(\alpha)$ and define $P_\alpha := \beta A_{\mathrm{supp}(\alpha')}\beta^{-1}$.

Proposition (minimal parabolic subgroup containing α)

- *P_α is well defined (i.e., does not depend on the choice of α').*
- *P_α is the smallest parabolic subgroup (by $\subseteq$) containing α.*

Sketch of proof:

- if α or α^{-1} is positive: use the previous Corollary; otherwise:
- use a power Garside structure such that the np-normal form of any conjugate in $RSSS_\infty(\alpha)$ is $x^{-1}y$ (where x, y are divisors of Δ^m)
- $\alpha', \alpha'' = (\alpha')^g \in RSSS_\infty(\alpha)$ implies $\left(\widetilde{C}^{(N)}_{\Delta^m}(\alpha')\right)^g = \widetilde{C}^{(N)}_{\Delta^m}(\alpha'')$
- $\left(\widetilde{C}^{(N)}_{\Delta^m}(\alpha')\right)^{-1}$ and $\left(\widetilde{C}^{(N)}_{\Delta^m}(\alpha'')\right)^{-1}$ are positive; use the Corollary (!)
- if $\alpha \in A_X$, constructing $\widehat{\alpha} \in A_X \cap RSSS_\infty(\alpha)$ yields $P_\alpha \subseteq A_X$

Intersection of parabolic subgroups

Definition ("size" of minimal containing parabolic)

Given α, pick $\alpha' \in RSSS_\infty(\alpha)$ and define $\varphi(\alpha) := \left|\Delta_{\mathsf{supp}(\alpha')}\right|$.

Lemma

$\varphi(\alpha)$ is well defined (i.e., does not depend on the choice of α').

Theorem (intersection of parabolic subgroups)

If P and Q are two parabolic subgroups of A_S, then $P \cap Q$ is also a parabolic subgroup of A_S.

Sketch of proof:

- take $\alpha \in P \cap Q$ for which the "size" $\varphi(\alpha)$ of P_α is maximal
- clearly $P_\alpha \subseteq P \cap Q$; to show equality: take arbitrary $w \in P_\alpha = A_Z$
  - $z_{P_w}(\Delta_Z)^m$ has has a positive conjugate if m is large
  - the support of this conjugate yields $z_{P_w} \in A_Z$, so $w \in A_Z$

Sketch of the details
└A few words on some of the proofs

Parabolic subgroups are closed under taking roots

Theorem (minimal parabolic subgroup of a power / root)

If $m \in \mathbb{Z} \setminus \{0\}$, then $P_{\alpha^m} = P_\alpha$.

Sketch of proof:

- assume $\alpha \in RSSS_\infty(\alpha)$ and $\alpha^m \in RSSS_\infty(\alpha^m)$
- if $x_i^{-1} y_i$ is the np-normal form of $\tilde{c}^i_{\Delta^N}(\alpha)$, then

$$\alpha^m = x_0^{-1} \cdots x_{m-1}^{-1} y_{m-1} \cdots y_0$$

- use an explicit formula for the conjugating element along the cycle of α from [Birman–G–GM 2007] to show that there is no cancellation between x_{m-1}^{-1} and y_{m-1}
- so α and α^m have the same support

Lattice of parabolic subgroups

Theorem (lattice of parabolic subgroups)

The poset of parabolic subgroups of A_S with respect to inclusion is a lattice.

Sketch of proof:

- greatest common lower bounds: intersection, by earlier theorem
- least common upper bounds:
  - only countably many parabolic subgroups T_i contain $P \cup Q$
  - consider $U_n = \bigcap_{i=0}^{n} T_i$; each U_n is a parabolic subgroup
  - the sequence $|\Delta_{U_i}|$ can only decrease finitely often
  - $\bigcap_{i=0}^{\infty} T_i = U_N$ is a parabolic subgroup