Pleasant actions on $\mathbb{Z}^d$

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Let $d \geq 5$. For every $0 < \delta < 1$ there exists $k(\delta)$ such that for every set $E \subset \mathbb{Z}^d$ of density δ there exists $k \leq k(\delta)$ with

$$k^2\mathbb{N} \subset \{x_1^2 + x_2^2 + \ldots + x_d^2 \,|\, (x_1, x_2, \ldots, x_d) \subset E - E\}.$$

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$\Gamma \curvearrowright \mathbb{Z}^d$ is **pleasant**, if for every $\mathbb{Z}^d \curvearrowright (X, \mu)$, and any $A \subset X$ with $\mu(A) > 0$ there exists $k = k(A) \geq 1$ such that for any vector $v \in \mathbb{Z}^d$ there exists $\gamma \in \Gamma$ with $\mu(A \cap \gamma(kv)A) > 0$.

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$\Gamma \curvearrowright \mathbb{Z}^3$ is pleasant $\Rightarrow$
$\forall E \subset \mathbb{Z}^3$ of positive density there exists $k \geq 1$ such that

$$k^2\mathbb{Z} = Q(k\mathbb{Z}^3) \subset Q(E - E).$$

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Notice that if $x + y^2 = m$ then $x + (y+n)^2 = m + 2yn + n^2$.

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$$s(n)(x, y) = (x - 2yn - n^2, y + n)$$

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There exists $s : \mathbb{Z}_{\geq 0} \to \Gamma$ with

- $\exists\, q_1(n, x_1, \ldots, x_d), q_2(n, x_1, \ldots, x_d), \ldots, q_d(n, x_1, \ldots, x_d) \in \mathbb{Z}[n, x_1, \ldots, x_d]$ with $s(n)(x_1, \ldots, x_d) = (q_1(n, x_1, \ldots, x_d), \ldots, q_d(n, x_1, \ldots, x_d))$.

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- $s(0) = Id_{\mathbb{Z}^d}$.

Polynomial walks on Γ and pleasantness of $\Gamma \curvearrowright \mathbb{Z}^d$

Main Theorem (Bulinski-Fish 2017)

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Let $p(y) \in \mathbb{Z}[y]$ with $p(0) = 0$ and $\deg p \geq 2$. Let $\Gamma \subset \{\mathbb{Z}^2 \to \mathbb{Z}^2\}$ be the polynomial maps preserving $x + p(y)$.

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Polynomial Bogoluboff's theorem, 2017

Let $E \subset \mathbb{Z}$ of positive density, and p is a polynomial with integer coefficients, $p(0) = 0$ and $deg(p) \geq 2$. Then there exists $k \geq 1$ such that $k\mathbb{Z} \subset (E - E) + p(E - E)$, e.g., the set $(E - E) + (E - E)^{25}$ contains an infinite periodic set.

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- **Claim.** $\mathbb{Z}^2 \curvearrowright (X, \mu)$ and $A \subset X$ with $\mu(A) > 0$. Then $\exists\ k \geq 1$ such that for every $(x, y) \in \mathbb{Z}^2\ \exists\ n \geq 1$ with $\mu(A \cap (kx - 2kyn - n^2, ky + n)A) > 0$.

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- By Spectral theorem, $\exists \eta$ non-negative measure on $\mathbb{T}^2$ (the dual of $\mathbb{Z}^2$) with

$$\mu(A \cap (a,b)A) = \int_{\mathbb{T}^2} \chi(a,b) d\eta(\chi) \text{ for all } (a,b) \in \mathbb{Z}^2.$$

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- Then by Weyl's equidistribution:

$$\frac{1}{N} \sum_{n=1}^{N} \chi((x - 2yn - n^2, y + n)) \to 0 \text{ for every irrational } \chi.$$

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- Therefore, for k large enough which depends on the atomic part of η "sitting" on rational characters, for all $(x, y) \in \mathbb{Z}^2$ we will have

$$\limsup_{N \to \infty} \int_{\mathbb{T}^2} \frac{1}{N} \sum_{n=1}^{N} \chi((kx - 2ky(kn) - (kn)^2, ky + kn)) d\eta(\chi) > 0.$$

Open Problem – Quantitative Pleasantness

Fish, 2017: For any $0 < \delta < 1$ there exists $k(\delta)$ such that for every $E_1, E_2 \subset \mathbb{Z}$ of densities at least δ there exists $k \leq k(\delta)$ with

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Is it true that if $\Gamma < GL_d(\mathbb{Z})$ is finitely generated by unipotent matrices, and $\Gamma \curvearrowright \mathbb{R}^d$ irreducibly, then for any $\delta > 0$ there exists $k(\delta)$ such that for every $\mathbb{Z}^d \curvearrowright (X, \mu)$, any $A \subset X$ with $\mu(A) \geq \delta$ there exists $k \leq k(\delta)$ such that for every $v \in \mathbb{Z}^d$:

$$\mu(A \cap \gamma(kv)A) > 0?$$