Endomorphisms, subgroups and quotients of totally disconnected locally compact groups

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Suppose G is any locally compact group. Let $G_0 \leq G$ be the connected component of the identity. We have the following short exact sequence:

$$G_0 \to G \to G/G_0$$

G/G_0 is a totally disconnected locally compact group.

Examples:

- Lie groups local fields (p-adic numbers or formal Laurant series over a finite fields);
- Automorphisms groups of trees, buildings, and other combinatorial structures.
- Discrete Groups and their completions.

Consider $\mathrm{BS}(m,n) = \langle a, t \mid ta^m t^{-1} = a^n \rangle$ acting on $\mathrm{BS}(m,n)/\langle a \rangle$. That is, there is a homomorphism

$$\varphi : \mathrm{BS}(m,n) \to \mathrm{Sym}(\mathrm{BS}(m,n)/\langle a \rangle).$$

The closure $G_{m,n} := \overline{\varphi(\mathrm{BS}(m,n))}$ with respect to the topology of pointwise convergence is a totally disconnected locally compact group.

Theorem (van Dantzig 1931)

A tdlc group has a basis at the identity consisting of compact open subgroups.

This means we can do a lot of counting. Eg if $U, V \leq G$ are compact and open and $\alpha \in \mathrm{End}(G)$, then the indices

$$[U : U \cap V] \text{ and } [\alpha(U) : \alpha(U) \cap U]$$

are both finite.

Definition (Willis 2015)

For and endomorphism α of a totally disconnected locally compact groups define the scale of α by

$$s_G(\alpha) = \min\{[\alpha(U) : U \cap \alpha(U)] : U \leq G \text{ compact open}\}.$$

If $s_G(\alpha) = [\alpha(U) : U \cap \alpha(U)]$ then we say U is tidy for α.

For $x \in G$ write $s_G(x)$ for the scale of the automorphism given by conjugation by x.

Examples of the scale for inner-automorphisms:

- (Glöckener 1998) P-adic lie groups: For $x \in G$ a p-adic Lie group let $\lambda_0, \ldots, \lambda_n$ be the eigenvalues of Ad_x. Then $s_G(x) = \prod_{|\lambda_i| \geq 1} |\lambda_i|$.
- (Elder, Willis 2013) $\mathrm{BS}(m,n)$ for m and n coprime: For $x = a^{p_0} t^{q_0} a^{p_1} t^{q_1} \cdots a^{p_k} t^{q_k} \in \mathrm{BS}(m,n)$ let $\rho = \sum q_i$. Then

$$s_{G_{m,n}}(x) = \begin{cases} \left(\dfrac{\mathrm{lcm}(m,n)}{|n|}\right)^{\rho} & \text{if } \rho \geq 0 \\ \left(\dfrac{\mathrm{lcm}(m,n)}{|m|}\right)^{|\rho|} & \text{if } \rho \leq 0 \end{cases}$$

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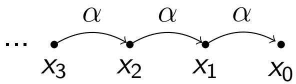
A sequence $(x_i)_{i \in \mathbb{N}}$ is an α-regressive trajectory at x_0 if $\alpha(x_{i+1}) = x_i$ for all $i \in \mathbb{N}$.



Define the antiparabolic subgroup of α to be

$$\mathrm{par}^{-}(\alpha) = \left\{ \begin{array}{c} \text{elements of } G \text{ which admit an } \alpha\text{-regressive} \\ \text{trajectory that is bounded} \end{array} \right\}.$$

Theorem (B., Glöckner, Tornier 2017)

Let G be tdlc and $\alpha \in \mathrm{End}(G)$. Suppose further that $H \leq G$ is closed and $\alpha(H) \leq H$. Let $\overline{\alpha} : G/H \to G/H$ be the map induced by α. Then:

1. *There is a compact open subgroup $U \leq G$ tidy for α such that $U \cap H$ is tidy for $\alpha|_H$. Furthermore*

$$s_H(\alpha|_H) \leq s_G(\alpha).$$

2. *If H is normal in G, then*

$$s_H(\alpha|_H) s_{G/H}(\overline{\alpha}) \text{ divides } s_G(\alpha).$$

3. *If $H \leq \mathrm{par}^-(\alpha)$ is a closed normal subgroup of G, then*

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For $\alpha \in \mathrm{End}(G)$ and $U \leq G$ compact open define

$$H_{top}(\alpha, U) := \lim_{n\to\infty} \frac{\log[U : \bigcap_{k=0}^{n} \alpha^{-k}(U)]}{n}$$

and

$$h_{top}(\alpha) = \sup\{H_{top}(\alpha, U) : U \leq G \text{ compact and open}\}.$$

Theorem (Giordano Bruno, Virili 2015)

$\bigcap\{U \leq G : U \textit{ is tidy for } \alpha\} = \{e\} \Leftrightarrow h_{top}(\alpha) = \log s_G(\alpha).$

Suppose G is a locally compact group, $\alpha \in \mathrm{End}(G)$, $H \leq G$ closed with $\alpha(H) \leq H$. Is it true that

$$h_{top}(\alpha) = h_{top}(\alpha|_H) + h_{top}(\overline{\alpha}),$$

where $\overline{\alpha} : G/H \to G/H$ is the continuous map induced by α?

Corollary (B., Glöckner, Tornier 2017)

Suppose G is totally disconnected and locally compact, $\alpha \in \mathrm{End}(G)$ such that $\mathrm{con}(\alpha)$ is closed, $H \trianglelefteq \mathrm{par}^-(\alpha)$ such that $\alpha(H) = H$. Then

$$h_{top}(\alpha) = h_{top}(\alpha|_H) + h_{top}(\overline{\alpha}).$$