Semigroup C^*-algebras

Nathan Brownlowe

The University of Sydney

28 November 2017

Motivation: We know what to do with groups

Let G be a discrete group.

For each $g \in G$, let $\lambda_g \in B(\ell^2(G))$ be given by $\lambda_g \delta_h = \delta_{gh}$. The **reduced group C^*-algebra of G** is

$$C_r^*(G) := C^*(\{\lambda_g : g \in G\}) \subseteq B(\ell^2(G)).$$

The **full group C^*-algebra** $C^*(G)$ is generated by a unitary representation $i_G\colon G \to C^*(G)$, and is universal in the following sense: if $u\colon G \to A$ is a unitary representation of G in a C^*-algebra A, then there is a homomorphism $C^*(G) \to A$ satisfying $i_G(g) \mapsto u_g$ for all $g \in G$.

Fact: $C^*(G) \cong C_r^*(G) \iff G$ is amenable $\iff C^*(G)$ is nuclear.

Question: How do we do this for semigroups?

Suppose S is a discrete semigroup with identity.

We want S to be **left cancellative**:

$$rs = rt \implies s = t \text{ for all } r, s, t \in S.$$

This ensures that $s \mapsto rs$ is injective for all $r \in S$, and that $\delta_s \mapsto \delta_{rs}$ extends to an isometry $V_r \in B(\ell^2(S))$.

The **reduced semigroup C^*-algebra of S** is

$$C_r^*(S) := C^*(\{V_s : s \in S\}) \subseteq B(\ell^2(S)).$$

A universal semigroup C^*-algebra?

Is there an analogue of the full group C^*-algebra $C^*(G)$ for semigroups?

The first guess is to build a C^*-algebra $C^*(S)$ which is universal for isometric representations of S.

In the case of $S = \mathbb{N}^2$, we have $C^*(\mathbb{N}^2)$ the universal C^*-algebra generated by two commuting isometries. But there is a problem: $\mathbb{N}^2$ is an amenable semigroup, but Murphy [4] showed that $C^*(\mathbb{N}^2)$ is not a nuclear C^*-algebra.

We look to $V\colon \mathbb{N}^2 \to B(\ell^2(\mathbb{N}^2))$ for more inspiration...

A universal semigroup C^*-algebra?

The isometries $V_{(1,0)}, V_{(0,1)} \in B(\ell^2(\mathbb{N}^2))$ satisfy

$$V_{(1,0)}^* V_{(0,1)} = V_{(0,1)} V_{(1,0)}^* \quad \text{and} \quad V_{(0,1)}^* V_{(1,0)} = V_{(1,0)} V_{(0,1)}^*.$$

We have

$$\begin{aligned} V_{(1,0)}^* V_{(0,1)} &= V_{(0,1)} V_{(1,0)}^* \\ &\implies V_{(1,0)} \big(V_{(1,0)}^* V_{(0,1)} \big) V_{(0,1)}^* = V_{(1,0)} \big(V_{(0,1)} V_{(1,0)}^* \big) V_{(0,1)}^* \\ &\implies V_{(1,0)} V_{(1,0)}^* V_{(0,1)} V_{(0,1)}^* = V_{(1,1)} V_{(1,1)}^*, \end{aligned}$$

which is reflecting the identity

$$\big((1,0) + \mathbb{N}^2\big) \cap \big((0,1) + \mathbb{N}^2\big) = (1,1) + \mathbb{N}^2.$$

Right LCM semigroups

Nica [5] made the key observation that a universal C^*-algebra of a semigroup should model the ideal structure of the semigroup. He introduced a class of semigroups called quasi-lattice ordered semigroups, and constructed a universal C^*-algebra to each such semigroup.

We go through the details for a larger class of semigroups:

A discrete left-cancellative semigroup S is called **right LCM** if the intersection of two principal right ideals is either empty or another principal right ideal. So for each $s, t \in S$ we have

$$sS \cap tS = \begin{cases} rS & \text{if } sS \cap tS \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

Right LCM semigroup C^*-algebras

We can use Li's theory [3] of semigroup C^*-algebras to associate C^*-algebras to right LCM semigroups.

The **full C^*-algebra** $C^*(S)$ of a right LCM semigroup S is

$$C^*\left(v_s \;\middle|\; v_s^* v_s = 1,\ v_s v_t = v_{st},\ v_s v_s^* v_t v_t^* = \begin{cases} v_r v_r^* & \text{if } sS \cap tS = rS \\ 0 & \text{if } sS \cap tS = \emptyset \end{cases}\right),$$

universal in the sense that for every family of isometries w_s, $s \in S$, in a C^*-algebra B satisfying the above relations, there is a homomorphism $\pi_w : C^*(S) \to B$ satisfying $\pi(v_s) = w_s$.

Examples of right LCM semigroups

We start with examples of quasi-lattice ordered semigroups:

- **Right-angled Artin monoids**: for $\Gamma = (V, E)$ a graph

$$A_\Gamma^+ := \langle v \in V \mid vw = wv \text{ if } (v, w) \in E \rangle^+.$$

  These include free monoids $\mathbb{F}_n^+$ and free abelian monoids $\mathbb{N}^k$.

- The $ax + b$**-semigroup** $\mathbb{N} \rtimes \mathbb{N}^\times$: the set $\mathbb{N} \times \mathbb{N}$ under

$$(m, a)(n, b) = (m + an, ab).$$

- For integers c and d with $cd > 1$, the **Baumslag–Solitar monoids**:

$$BS(c, d)^+ = \langle a, b \mid ab^c = b^d a \rangle^+.$$

More examples of right LCM semigroups

- **Graph products**: for $\Gamma = (V, E)$ a graph, and $\{S_v : v \in V\}$ a family of right LCM semigroups, the graph product S_Γ is the quotient of $\bigoplus_{v \in V} S_v$ by the congruence generated by the relation $\{(st, ts) : s \in S_v,\ t \in S_w, (v, w) \in E\}$.

- The $ax + b$-**semigroup** $R \rtimes R^\times$ for R a Principal Ideal Domain.

- **Binary adding machine**: For $\mathbb{Z} = \langle a \rangle$ we set

$$a \cdot 0 = 1 \qquad a|_0 = e$$
$$a \cdot 1 = 0 \qquad a|_1 = a,$$

  and extend this to words in $\{0, 1\}^*$ via

$$a \cdot iw = (a \cdot i)(a|_i \cdot w).$$

  Then $\{0, 1\}^* \times \mathbb{Z}$ under $(w, a^m)(z, a^n) = (w(a^m \cdot z), a^m|_z a^n)$ is right LCM.

The other Baumslag–Solitar monoid

Let $c, d \geq 1$ and

$$S := BS(c, -d)^+ = \langle a, b \mid ab^c = b^{-d} a \rangle^+.$$

Elements $s \in S$ have a normal form

$$s = b^{i_1} a b^{i_2} a \cdots b^{i_\ell} a b^p \quad \text{with } 0 \leq i_j \leq d-1,\ p \in \mathbb{Z},$$

and we get maps $\omega \colon S \to \mathbb{F}_d^+ = \langle a, ba, \ldots, b^{d-1}a \rangle$, $\ell \colon S \to \mathbb{N}$, and $p \colon S \to \mathbb{Z}$ given by

$$\omega(s) = b^{i_1} a b^{i_2} a \cdots b^{i_\ell} a, \quad \ell(s) = \ell, \quad \text{and} \quad p(s) = p.$$

The other Baumslag–Solitar monoid

For $s, t \in S$ we have

- $s \in tS$ if $\omega(s) \in \omega(t)\mathbb{F}_d^+$ and $\ell(s) > \ell(t)$;
- $s \in tS$ if $\omega(s) \in \omega(t)\mathbb{F}_d^+$, $\ell(s) = \ell(t)$, and $p(s) \geq p(t)$; and
- $sS \cap tS = \emptyset$ if $\omega(s) \notin \omega(t)\mathbb{F}_d^+$.

It follows that elements of $BS(c, -d)^+$ have disjoint principal right ideals, or are comparable, in the sense that one principal right ideal is contained in the other.

So $BS(c, -d)^+$ is right LCM.

The other Baumslag–Solitar monoid

$BS(c,-d)^+$ is not quasi-lattice ordered when $c>1$:

Using the partial order given by $g \leq h \iff h \in gS$, we have (G,S) quasi-lattice ordered if every $g \in G$ with an upper bound in S has a least upper bound in S.

Using that $b^d ab^c = a$ we have

$$\begin{aligned} a &= (aba^{-1})ab^{-1} = (aba^{-1})b^d ab^{c-1} \in aba^{-1}S \\ ab^{-1} &= (aba^{-1})ab^{-2} = (aba^{-1})b^d ab^{c-2} \in aba^{-1}S \\ ab^{-2} &= (aba^{-1})ab^{-3} = (aba^{-1})b^d ab^{c-3} \in aba^{-1}S \\ &\vdots \end{aligned}$$

And obviously $ab^{-n} = ab^{-(n+1)}b \in ab^{-(n+1)}S$ for each n. So

$$aba^{-1} \leq \qquad \cdots \leq ab^{-2} \leq ab^{-1} \leq a,$$

which says that aba^{-1} has an upper bound in S, but no least upper bound in S.

- J. Crisp and M. Laca, *Boundary quotients and ideals of Toeplitz C^*-algebras of Artin groups*, J. Funct. Anal. **242** (2007), 127–156.
- M. Laca and I. Raeburn, *Phase transition on the Toeplitz algebra of the affine semigroup over the natural numbers*, Adv. Math., **225** (2010), 643–688.
- X. Li, *Semigroup C^*-algebras and amenability of semigroups*, J. Funct. Anal. **262** (2012), 4302–4340.
- G. J. Murphy, *C^*-algebras generated by commuting isometries*, Rocky Mountain J. Math. **26** (1996), 237–267.
- A. Nica, *C^*-algebras generated by isometries and Wiener-Hopf operators*, J. Operator Theory **27** (1992), 17–52.
- J. Spielberg, *C^*-algebras for categories of paths associated to the Baumslag–Solitar groups*, J. Lond. Math. Soc. **86**, (2012), 317–360.

Thanks!