

The classification of subgroups of a group which are invariant under the action by conjugation of a fixed subgroup

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November 28, 2017

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Introduction

In 1830, E. Galois invented at the age of 19 the concept of

- ▶ **group**

and defined the notions of

- ▶ **normal subgroup**

- ▶ **solvable group.**

He did this in order to prove his famous result that the roots of a polynomial with coefficients in a field can be obtained by adjoining roots of unity to the ground field if and only if the Galois group of the polynomial is a solvable group.

Ever since then, the topic group theory has to a major extent revolved around

- ▶ **K -normality: the notion that a subgroup H of a group G is normalized by a subgroup K of G , i.e. H is left invariant under conjugation by the elements of K (a generalization of the notion of normal subgroup)**
- ▶ **the notion of mixed commutator subgroup $[K, H]$ and its use in making definitions and constructions** (which Galois did in defining solvable group).

This talk uses mixed commutator subgroups to describe the set of all subgroups of a group G which are normalized by a fixed subgroup K of G . The main result will be the following:

Let G be a group. Fix a subgroup K of G . A subgroup F of G is called **K-fundamental** or simply **fundamental**, if there is a subgroup $L \leq G$ such that $F = [K, L]$. Let \mathcal{F} denote the set of all fundamental subgroups of G . Let

- ▶ **BSand(F) = bubble sandwich at F**

which will be defined later. Then the

- ▶ **set of K-normal subgroups of G = $\bigcup_{F \in \mathcal{F}} \text{BSand}(F)$.**

After describing in detail the sandwich classification theorem above, we give a historical account of the development of sandwich classification in group theory, beginning with results of C. Jordan in 1870 classifying the normal subgroups of the general linear group $GL_n(F_p)$ of the prime field F_p where p is a prime number. We close the talk with some open problems.

Language, notation and the sandwich classification theorem

$x, y \in G$, define the **commutator** $[x, y] = xyx^{-1}y^{-1}$

$K, L \leq G$, define the **mixed commutator subgroup** $[K, L] =$ subgroup of G generated by all commutators $[k, \ell]$ such that $k \in K, \ell \in L$.

$F \leq G$ is called **K-fundamental**, if there is an $L \leq G$ such that $F = [K, L]$.

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F is K -normal (as well as L -normal).

$\mathcal{F} =$ set of all fundamental subgroups of G .

$\mathcal{F}^+ = \text{Subgr}(G)$

$X \subseteq \text{Subgr}(G)$ is called a **sandwich**, if X has a unique **smallest member** A and a unique **largest member** B such that $X = \{H \leq G \mid A \leq H \leq B\}$. We denote X by

▶ **Sand(A,B).**

$X \subseteq \text{Subgp}(G)$ is called a **generalized sandwich**, if X has a unique **largest member** C such that such that if $B \in X$ then $\text{Sand}(B, C) \subseteq X$.

Theorem (Sandwich Classification Theorem)

Let $F \in \mathcal{F}$ (= set of all fundamental subgroups of G), then

- ▶ $BSand(F) = \{L \leq G \mid [K, L] = F\}$ is a generalized sandwich called the bubble sandwich at F . We let $C(F)$ denote the unique maximal member of $BSand(F)$. Both F and $C(F)$ are K -normal.
- ▶ $\mathcal{F}^+ = \bigcup_{F \in \mathcal{F}} BSand(F)$.

Theorem (Sandwich Classification Theorem)

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- ▶ $\mathcal{F}^+ = \bigcup_{F \in \mathcal{F}} BSand(F)$.
- ▶ $bSand(F) = \{L \leq G \mid [K, L] = F, L \text{ } K\text{-normal}\}$ is a generalized sandwich called the K -normal bubble sandwich at F . Its unique maximal member is the $C(F)$ above.
- ▶ $K\text{-Nor}(\mathcal{F}^+) = \bigcup_{F \in \mathcal{F}} bSand(F)$.

Theorem (Generalized Sandwich Classification Theorem)

If $L \leq G$, let $D(L) = [K, L]$. Let $X \subseteq \mathcal{F}^+$ such that for each $L \in X$, $C(D(L)) \in X$. Then

- ▶ for each $L \in X$, $BSand(D(L)) \cap X$ is a generalized sandwich with unique maximal element $C(D(L))$.
- ▶ $X = \bigcup_{F=D(L), L \in X} (BSand(F) \cap X)$.

K-nilpotency

$K, L \leq G$. Define

$$\blacktriangleright D_K^0(L) = L.$$

For $n > 0$, define recursively

$$\blacktriangleright D_K^n(L) = D_K^{n-1}(L).$$

Since K is and will be fixed, we let

$$\blacktriangleright D^n(L) = D_K^n(L).$$

The series

$$\blacktriangleright L = D^0(L) \geq D^1(L) \geq \dots \geq D^n(L) \geq \dots$$

is called the **descending K-central series** of L .

Theorem

The descending K -central series is a normal series and the action of K on each consecutive quotient $D^n(L)/D^{n+1}(L)$ is trivial. Moreover, if L normalizes K then each group $D^n(L)$ is normal in L .

L is called **K -nilpotent**, if from some n , $D^n(L) = D^{n+1}(L)$. The smallest n such that the above holds is called the **K -nilpotent class** of L . If $D^n(L) = 1$ for some n , we say that L is **absolutely K -nilpotent**.

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L is called **K -perfect**, if $L = D(L)$. Clearly K -perfect subgroups are fundamental subgroups.

The descending K -central series stabilizes precisely when $D^n(L)$ becomes K -perfect.

The exact opposite of a K -perfect subgroup is a **K-imperfect subgroup**. It is a subgroup that is not fundamental, i.e. L is K -imperfect, if there is no subgroup L' such that $D(L') = L$. We shall encounter these subgroups later on.

Subgr(G) as a forest and subforests

Let Σ be a partially ordered set with partial ordering given by \leq . If $x \in \Sigma$, let $\Sigma(< x) = \{y \in \Sigma \mid y < x\}$.

- ▶ Σ is called a **forest**, if for any element $x \in \Sigma$ such that $\Sigma(< x)$ is nonempty, the set $\Sigma(< x)$ contains a unique largest element, which is denoted by $glb(x)$ and called the **greatest lower bound** of x .

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Σ a forest. $x > y$ is called an **elementary path**, if $y = glb(x)$.

$x = x_0 > x_1 > \dots > x_n$ is called a **path**, if each $x_i > x_{i+1}$ is an elementary path. n is called the **length** of the path.

If $x \neq y$ then there is at most one path from x to y . Of course, if there is a path from x to y then there is no path from y to x .

$x, y \in \Sigma$ are called **connected**, if either $x = y$ or there is a point z in X and paths from x to z and y to z .

A connected subset of Σ is called a **tree**. Every tree is contained in a maximal tree.

Σ is the disjoint union of its maximal trees.

$x, y \in \Sigma$. Define the **distance**

$$d(x, y) = \begin{cases} \infty, & \text{if } \nexists \text{ path from } x \text{ to } y \text{ or } y \text{ to } x \\ n, & \text{if } \exists \text{ a path from } x \text{ to } y \text{ or } y \text{ to } x \text{ of length } n \\ 0, & \text{if } x = y. \end{cases}$$

Define the **height** $ht(\Sigma) =$ the smallest number, possibly infinity, such that $d(x, y) \leq n \forall x, y \in \Sigma$.

$x \in \Sigma$, let $D(x) = \text{glb}(x)$

A subset $X \subseteq \Sigma$ is called a **subforest**, if it satisfies the property that $x \notin X \implies D(x) \notin X$

A subforest is automatically a forest

X is a subforest of Σ , it is **not necessary** that if T is a tree in Σ , then $T \cap X$ is a tree in X .

In particular, if $x, y \in X$ then the distance between them in X can be infinite, while the distance between them in Σ is finite.

The forest structure on $\text{Subgr}(G)$

Recall:

- ▶ $\mathcal{F}^+ = \text{Subgp}(G)$
- ▶ $\mathcal{F} =$ all fundamental subgroups of G

The subgroups of \mathcal{F}^+ which are missing in \mathcal{F} are precisely the K -imperfect subgroups.

Partially order \mathcal{F}^+ , by defining $M \geq L \iff$ there is a descending K -central sequence $D^i(M)$ such that $D^n(M) = L$ for some n .

Lemma

The partial ordering above of \mathcal{F}^+ is a forest ordering such that \mathcal{F} is a subforest. Moreover, $ht(\mathcal{F}^+) \leq ht(\mathcal{F}) + 1$.

Call a subforest X of \mathcal{F}^+ **good**, if it has the property that if $L \in X$ then $C(D(L)) \in X$.

Call a subforest X of \mathcal{F}^+ **closed**, if it has the property that $L \in X \implies D(L) \in X$

By the Generalized Sandwich Classification Theorem, if X is good, then it satisfies sandwich classification, namely

$$\blacktriangleright X = \bigcup_{F=D(L), L \in X} (BSand(F) \cap X)$$

where each $BSand(F) \cap X$ is a generalized sandwich.

Theorem (Main Theorem)

Suppose X is a subforest of \mathcal{F}^+ which is closed and good. If $ht(X) = n$ is finite then the following holds:

- ▶ *the K -perfect subgroups are the subgroups of height 0.*
- ▶ *every maximal tree contains a unique subgroup of height 0 and there is a one to one correspondence between maximal trees and their subgroups of height 0.*
- ▶ *every member L of X is a member of exactly one maximal tree T and the nilpotent class (L) is less than or equal to the nilpotent class (T) is less than or equal to the nilpotent class $X = \text{supremum of the nilpotent classes of the trees.}$*
- ▶ *X and its maximal trees have compatible sandwich classifications.*

Theorem

Suppose K is K -perfect. Let X be a closed, good subforest of \mathcal{F}^+ .
Then $ht(X) \leq 1$ and each sandwich
 $BS(F) \cap X = Sand(F, C(F)) \cap X$.