The classification of subgroups of a group which are invariant under the action by conjugation of a fixed subgroup

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Introduction

In 1830, E. Galois invented at the age of 19 the concept of

- **group**

and defined the notions of

- **normal subgroup**
- **solvable group.**

He did this in order to prove his famous result that the roots of a polynomial with coefficients in a field can be obtained by adjoining roots of unity to the ground field if and only if the Galois group of the polynomial is a solvable group.

Ever since then, the topic group theory has to a major extent revolved around

- K**-normality: the notion that a subgroup** H **of a group** G **is normalized by a subgroup** K **of** G**, i.e.** H **is left invariant under conjugation by the elements of** K (a generalization of the notion of normal subgroup)
- **the notion of mixed commutator subgroup** $[K, H]$ **and its use in making definitions and constructions** (which Galois did in defining solvable group).

This talk uses mixed commutator subgroups to describe the set of all subgroups of a group G which are normalized by a fixed subgroup K of G. The main result will be the following:

Let G be a group. Fix a subgroup K of G. A subgroup F of G is called **K-fundamental** or simply **fundamental**, if there is a subgroup $L \leq G$ such that $F = [K, L]$. Let $\mathcal{F}$ denote the set of all fundamental subgroups of G. Let

- **BSand(F) = bubble sandwich at F**

which will be defined later. Then the

- **set of K-normal subgroups of G** $= \bigcup_{F \in \mathcal{F}}$ **BSand(F).**

After describing in detail the sandwich classification theorem above, we give a historical account of the development of sandwich classifiction in group theory, beginning with results of C. Jordan in 1870 classifying the normal subgroups of the general linear group $GL_n(F_p)$ of the prime field F_p where p is a prime number. We close the talk with some open problems.

Language, notation and the sandwich classification theorem

$x, y \in G$, define the **commutator** $[x, y] = xyx^{-1}y^{-1}$

$K, L \leq G$, define the **mixed commutator subgroup** $[K, L] =$ subgroup of G generated by all commutators $[k, \ell]$ such that $k \in K, \ell \in L$.

$F \leq G$ is called **K-fundamental**, if there is an $L \leq G$ such that $F = [K, L]$.

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$F \leq G$ is called **K-fundamental**, if there is an $L \leq G$ such that $F = [K, L]$.

F is K-normal (as well as L-normal).

$\mathcal{F}$ = set of all fundamental subgroups of G.

$\mathcal{F}^+ = Subgr(G)$

$X \subseteq Subgr(G)$ is called a **sandwich**, if X has a unique **smallest member** A and a unique **largest member** B such that $X = \{H \leq G \mid A \leq H \leq B\}$. We denote X by

- **Sand(A,B)**.

$X \subseteq Subgp(G)$ is called a **generalized sandwich**, if X has a unique **largest member** C such that such that if $B \in X$ then $Sand(B, C) \subseteq X$.

Theorem (Sandwich Classification Theorem)

Let $F \in \mathcal{F}$ (= set of all fundamental subgroups of G), then

- $BSand(F) = \{L \leq G \mid [K, L] = F\}$ *is a generalized sandwich called the bubble sandwich at F. We let $C(F)$ denote the unique maximal member of $BSand(F)$. Both F and $C(F)$ are K-normal.*
- $\mathcal{F}^+ = \bigcup_{F \in \mathcal{F}} BSand(F)$.

Theorem (Sandwich Classification Theorem)

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- $\mathcal{F}^+ = \bigcup_{F \in \mathcal{F}} BSand(F)$.
- *bSand(F)*$= \{L \leq G \mid [K, L] = F, L \quad K{-}normal\}$ *is a generalized sandwich called the K-normal bubble sandwich at F.It's unique maximal member is the $C(F)$ above.*
- $K\text{-}Nor(\mathcal{F}^+) = \bigcup_{F \in \mathcal{F}} bSand(F)$.

Theorem (Generalized Sandwich Classification Theorem)

If $L \leq G$, let $D(L) = [K, L]$. Let $X \subseteq \mathcal{F}^+$ such that for each $L \in X$, $C(D(L)) \in X$. Then

- *for each $L \in X$, $BSand(D(L)) \cap X$ is a generalized sandwich with unique maximal element $C(D(L)$.*
- $X = \bigcup_{F=D(L), L \in X}(BSand(F) \cap X$.

K-nilpotency

$K, L \leq G$. Define

- $D_K^0(L) = L$.

For $n > 0$, define recursively

- $D_K^n(L) = D_K^{n-1}(L)$.

Since K is and will be fixed, we let

- $D^n(L) = D_K^n(L)$.

The series

- $L = D^0(L) \geq D^1(L) \geq ..... \geq D^n(L) \geq .....$

is called the **descending K-central series** of L.

Theorem

The descending K-central series is a normal series and the action of K on each consecutive quotient $D^n(L)/D^{n+1(L)}$ is trivial. Moreover, if L normalizes K then each group $D^n(L)$ is normal in L.

L is called **K-nilpotent**, if from some n, $D^n(L) = D^{n+1}(L)$. The smallest n such that the above holds is called the **K-nilpotent class** of L. If $D^n(L) = 1$ for some n, we say that L is **absolutely K-nilpotent**.

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L is called **K-perfect**, if $L = D(L)$. Clearly K- perfect subgroups are fundamental subgroups.

The descending K-central series stabilizes precisely when $D^n(L)$ becomes K-perfect.

The exact opposite of a K-perfect subgroup is a **K-imperfect subgroup**. It is a subgroup that is not fundamental, i.e.L is K-imperfect, if if there is no subgroup L' such that $D(L') = L$.We shall encounter these subgroups later on.

Subgr(G) as a forest and subforests

Let Σ be a partially ordered set with partial ordering given by $\leq$. If $x \in \Sigma$, let $\Sigma(< x) = \{y \in \Sigma \mid y < x\}$.

- Σ is called a **forest**, if for any element $x \in \Sigma$ such that $\Sigma(< x)$ is nonempty, the set $\Sigma(< x)$ contains a unique largest element, which is denoted by $glb(x)$ and called the **greatest lower bound** of x.

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Σ a forest. $x > y$ is called an **elementary path**, if $y = glb(x)$.

$x = x_0 > x_1 > ...... > x_n$ is called a **path**, if each $x_i > x_{i+1}$ is an elementary path. n is called the **length** of the path.

If $x \neq y$ then there is at most one path from x to y. Of course, if there is a path from x to y then there is no path from y to x.

$x, y \in \Sigma$ are called **connected**, if either $x = y$ or there is a point z in X and paths from x to z and y to z.

A connected subset of Σ is called a **tree**. Every tree is contained in a maximal tree.

Σ is the disjoint union of its maximal trees.

$x, y \in \Sigma$. Define the **distance**

$$d(x,y) = \begin{cases} \infty, & \text{if } \nexists \text{ path from } x \text{ to } y \text{ or } y \text{ to } x \\ n, & \text{if } \exists \text{ a path from } x \text{ to } y \text{ or } y \text{ to } x \text{ of length } n \\ 0, & \text{if } x = y. \end{cases}$$

Define the **height** $ht(\Sigma) =$ the smallest number, possibly infinity, such that $d(x,y) \leq n \forall x, y \in \Sigma$.

$x \in \Sigma$, let $D(x) = glb(x)$

A subset $X \subseteq \Sigma$ is called a **subforest**, if it satisfies the property that $x \notin X \Longrightarrow D(x) \notin X$

A subforest is automatically a forest

X is a subforest of Σ, it is **not necessary** that if T is a tree in Σ, then $T \cap X$ is a tree in X.

In particular, if $x, y \in X$ then the distance between them in X can be infinite, while the distance between them in Σ is finite.

The forest structure on Subgr(G)

Recall:

- $\mathcal{F}^+$=*Subgp*(G)
- $\mathcal{F}$= *all fundamental subgroups of G*

The subgroups of $\mathcal{F}^+$ which are missing in $\mathcal{F}$ are precisely the K-imperfect subgroups.

Partially order $\mathcal{F}^+$, by defining $M \geq L \iff$ there is a descending K-central sequence $D^i(M)$ such that $D^n(M) = L$ for some n.

Lemma

The partial ordering above of $\mathcal{F}^+$ is a forest ordering such that $\mathcal{F}$ is a subforest. Moreover, $ht(\mathcal{F}^+) \leq ht(\mathcal{F}) + 1$.

Call a subforest X of $\mathcal{F}^+$ **good**, if it has the property that if $L \in X$ then $C(D(L)) \in X$.

Call a subforest X of $\mathcal{F}^+$ **closed**, if it has the property that $L \in X \Longrightarrow D(L) \in X$

By the Generalized Sandwich Classification Theorem, if X is good, then it satisfies sandwich classification, namely

- $X = \bigcup_{F=D(L), L \in X} (BSand(F) \cap X)$

where each $BSand(F) \cap X$ is a generalized sandwich.

Theorem (Main Theorem)

Suppose X is a subforest of $\mathcal{F}^+$ which is closed and good. If $ht(X) = n$ is finite then the following holds:

- *the K-perfect subgroups are the subgroups of height 0.*
- *every maximal tree contains a unique subgroup of height 0 and there is a one to one correspondence between maximal trees and their subgroups of height 0.*
- *every member L of X is a member of exactly one maximal tree T and the nilpotent class (L) is less than or equal to the nilpotent class (T) is less than or equal to the nilpotent class $X =$ supremum of the nilpotent classes of the trees.*
- *X and its maximal trees have compatible sandwich classifications.*

Theorem

Suppose K is K-perfect. Let X be a closed, good subforest of $\mathcal{F}^+$*. Then* $ht(X) \leq 1$ *and each sandwich* $BS(F) \cap X = Sand(F, C(F)) \cap X$*.*