Simple graph algebras

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(Joint work with Nathan Brownlowe and Aidan Sims)

Simple C^*-algebras

Definition

A **C^*-algebra** is a Banach $*$-algebra A over $\mathbb{C}$ which satisfies the **C^*-identity**: for all $a \in A$,

$$\|a^* a\| = \|a\|^2.$$

An **ideal** I of A is a closed $*$-subalgebra of A such that

$$ax \in I \quad \text{and} \quad xa \in I,$$

for all $a \in A$ and $x \in I$.

We say that A is **simple** if its only ideals are itself and the trivial ideal $\{0\}$.

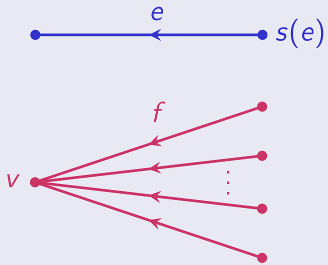
Definition (Kumjian–Pask–Raeburn–Renault 1997)

Suppose that $E = (E^0, E^1, r, s)$ is a row-finite, source-free directed graph (meaning that $0 < |r^{-1}(v)| < \infty$ for all $v \in E^0$), and A is a C^*-algebra. We say that

$$\{p_v : v \in E^0\}, \{s_e : e \in E^1\} \subseteq A$$

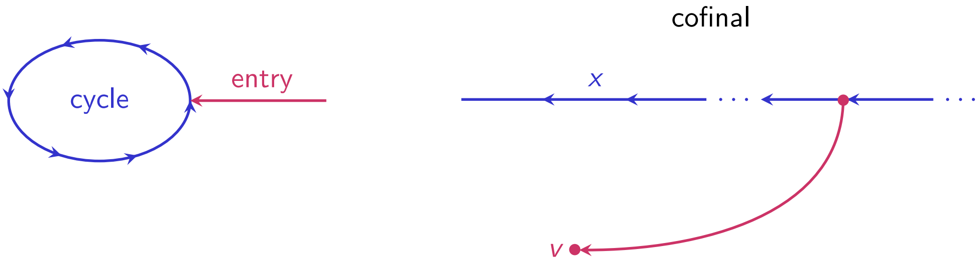
is a **Cuntz–Krieger E-family** if

- $p_v = p_v^* = p_v^2$;
- $u \neq v \implies p_u p_v = 0$;
- $s_e s_e^* s_e = s_e$;
- $s_e^* s_e = p_{s(e)}$; and
- $p_v = \sum_{f \in r^{-1}(v)} s_f s_f^*$.



There is a C^*-algebra $C^*(E)$ that is universal for Cuntz–Krieger E-families.

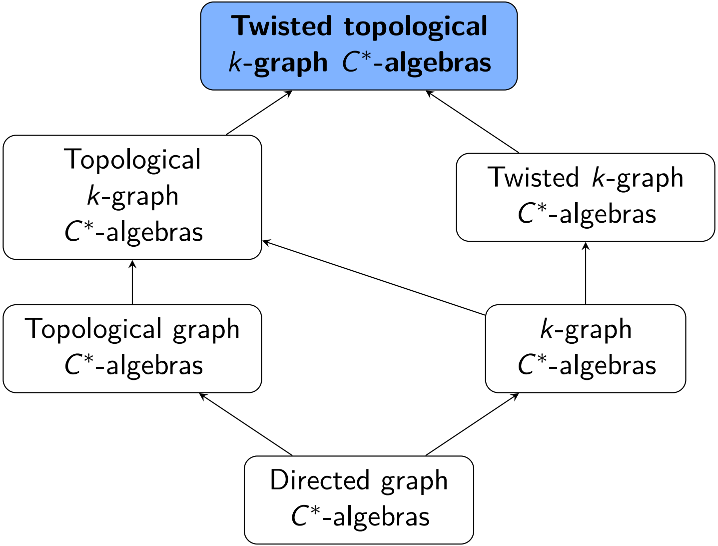
Characterising simplicity of $C^*(E)$ in terms of E



Theorem (Bates–Pask–Raeburn–Szymański 2000)

Let E be a row-finite, source-free directed graph. Then $C^*(E)$ is simple if and only if E is cofinal and every cycle in E has an entry.

Directed graph C^*-algebras and their generalisations



Topological k-graphs

Definition (Yeend 2006)

A **topological k-graph** (Λ, d) consists of a small category $\Lambda = (\mathrm{Obj}(\Lambda), \mathrm{Mor}(\Lambda), r, s, \circ)$ and a continuous functor $d\colon \Lambda \to \mathbb{N}^k$, called the **degree map**, such that

(i) $\mathrm{Obj}(\Lambda)$ and $\mathrm{Mor}(\Lambda)$ are both second-countable, locally compact, Hausdorff spaces;

(ii) the **range** and **source** maps $r, s\colon \mathrm{Mor}(\Lambda) \to \mathrm{Obj}(\Lambda)$ are continuous, and s is a local homeomorphism;

(iii) composition of morphisms (or **paths**) is continuous and open; and

(iv) the **unique factorisation property** holds: for all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$, there exists a unique composable pair $(\mu, \nu) \in \Lambda \times_c \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = m$, and $d(\nu) = n$.

For $n \in \mathbb{N}^k$, we define $\Lambda^n := d^{-1}(n)$. We denote the **infinite-path space** of Λ by Λ^∞.

Topological k-graph groupoids

We define the **path groupoid** of Λ to be the set

$$\mathcal{G}_\Lambda := \{(\mu x,\, d(\mu) - d(\nu),\, \nu x) \,:\, \mu, \nu \in \Lambda,\, x \in \Lambda^\infty\} \subseteq \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty,$$

with multiplication given by $(x, m, y)(y, n, z) := (x, m + n, z)$, inversion given by $(x, m, y)^{-1} := (y, -m, x)$, and unit space $\mathcal{G}_\Lambda^{(0)} \cong \Lambda^\infty$.

We give $\mathcal{G}_\Lambda$ a second-countable, locally compact, Hausdorff topology that incorporates the topology on Λ. Furthermore, $\mathcal{G}_\Lambda$ is an étale groupoid.

Definition

A **continuous $\mathbb{T}$-valued 2-cocycle** on $\mathcal{G}_\Lambda$ is a continuous map $\sigma\colon \mathcal{G}_\Lambda^{(2)} \to \mathbb{T}$ satisfying

(C1) $\sigma(\alpha,\beta)\,\sigma(\alpha\beta,\gamma) = \sigma(\alpha,\beta\gamma)\,\sigma(\beta,\gamma)$; and

(C2) $\sigma(\alpha, s(\alpha)) = \sigma(r(\alpha),\alpha) = 1$.

The space $C_c(\mathcal{G}_\Lambda,\sigma)$ is a $*$-algebra under the operations

$$(fg)(\gamma) := \sum_{\alpha\beta=\gamma} \sigma(\alpha,\beta) f(\alpha) g(\beta) \quad \text{and} \quad f^*(\gamma) := \overline{\sigma(\gamma,\gamma^{-1}) f(\gamma^{-1})}.$$

The **twisted topological k-graph C^*-algebra** $C^*(\mathcal{G}_\Lambda,\sigma)$ is defined to be the completion of $C_c(\mathcal{G}_\Lambda,\sigma)$ under the maximal C^*-norm.

The **isotropy** of $\mathcal{G}_\Lambda$ is the set

$$\mathrm{Iso}(\mathcal{G}_\Lambda) = \{(x, m, y) \in \mathcal{G}_\Lambda : x = y\}.$$

We define $\mathcal{I}_\Lambda$ to the topological interior of $\mathrm{Iso}(\mathcal{G}_\Lambda)$, and it is a subgroupoid of $\mathcal{G}_\Lambda$.

Lemma (A–Brownlowe–Sims 2017)

Define $\mathrm{Per}(\Lambda) := \{m \in \mathbb{Z}^k : (x, m, x) \in \mathcal{G}_\Lambda$ *for all* $x \in \Lambda^\infty\}$. *Then* $\mathcal{I}_\Lambda \cong \Lambda^\infty \times \mathrm{Per}(\Lambda)$.

Proposition (A–Brownlowe–Sims 2017)

Every continuous 2-cocycle on $\mathcal{G}_\Lambda$ *is cohomologous to a 2-cocycle* σ *on* $\mathcal{G}_\Lambda$ *that is constant on* $\mathcal{I}_\Lambda$, *in the sense that there is a bicharacter* ω *of* $\mathrm{Per}(\Lambda)$ *such that*

$$\sigma((x, m, x)(x, n, x)) = \omega(m, n),$$

for all $x \in \Lambda^\infty$, *and* $m, n \in \mathrm{Per}(\Lambda)$. *We write* $\sigma|_{\mathcal{I}_\Lambda^{(2)}} = 1_{\Lambda^\infty} \times \omega$.

Theorem (A–Brownlowe–Sims 2017)

Let Λ be a cofinal, proper, source-free topological k-graph. Suppose that σ is a continuous 2-cocycle on $\mathcal{G}_\Lambda$ and ω is a bicharacter of $\mathrm{Per}(\Lambda)$ such that $\sigma|_{\mathcal{I}_\Lambda^{(2)}} = 1_{\Lambda^\infty} \times \omega$. There is an action

$$\theta \colon \mathcal{G}_\Lambda/\mathcal{I}_\Lambda \curvearrowright \Lambda^\infty \times \widehat{Z_\omega}$$

given by

$$\theta_{[\gamma]}(s(\gamma), \phi) = (r(\gamma), \chi^\sigma_{[\gamma]}\phi).$$

The twisted topological k-graph C^*-algebra $C^*(\mathcal{G}_\Lambda, \sigma)$ is simple if and only if the action θ is minimal (meaning that every orbit is dense in $\Lambda^\infty \times \widehat{Z_\omega}$).

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Thanks!