On $\mathcal{F}(1,2)$-invariant Graphs

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Introduction

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A k-factor of X is a spanning subgraph Y such that each vertex has valency k in Y.

A vertex-transitive graph X is said to be $\mathcal{F}(a_1, a_2, \ldots, a_t)$-invariant if there is a partition of the edge set $E(X)$ into a_i-factors, $1 \leq i \leq t$, such that $\mathrm{Aut}(X)$ preserves the partition.

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Given a vertex-transitive graph X, let $E_1, E_2, \ldots, E_t$ be the edge orbits under $\mathrm{Aut}(X)$. Each E_i induces a spanning subgraph of some valency k_i.

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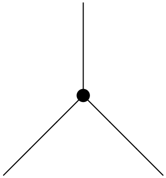
A second way to obtain examples is to start with an arc-transitive trivalent graph X and carry out neighbourhood expansion as shown in the next picture.

Examples

Replace each vertex of X with a triangle and join the vertices of the triangles according to the original edges.



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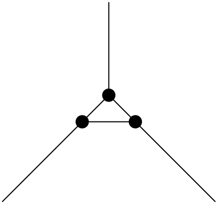

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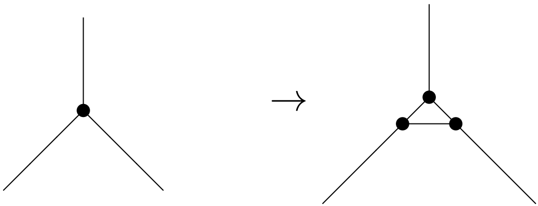

The resulting graph is vertex-transitive because X is arc-transitive. The 2-factor made up of triangles is invariant because the edges between triangles do not belong to any triangles. So the resulting graph is $\mathcal{F}(1,2)$-invariant.

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The subsequent material is work being carried out jointly by myself and Don Kreher and Afsaneh Khodadadpour.

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We decided to approach the problem by considering the number of cycles comprising F.

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Proposition. *If X is an order n $\mathcal{F}(1,2)$-invariant graph such that the 2-factor is a Hamilton cycle, then $\mathrm{Aut}(X) \leq D_n$, where D_n is the dihedral group of degree n and order $2n$.*

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The proposition follows from the fact that $\mathrm{Aut}(X)$ maps F to itself and the automorphism group of an n-cycle is D_n.

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The circulant graph of even order $n > 6$ with connection set $\{1, n/2, n-1\}$ is $\mathcal{F}(1, 2)$-invariant because the edges of F belong to a single 4-cycle and the edges of the 1-factor belong to two 4-cycles. The figure on the next slide indicates clearly why this is the case.

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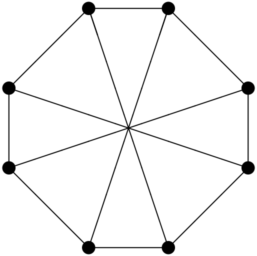

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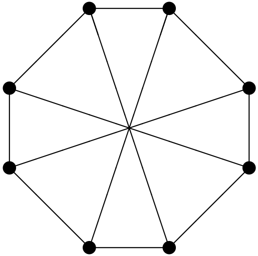

So we know precisely which circulant graphs are $\mathcal{F}(1,2)$-invariant for $m = 1$.

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Now take the reflection

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The group $\langle \rho^2, \tau \rangle$ is transitive, has order n and is isomorphic to $D_{n/2}$. Thus, X is a Cayley graph on $D_{n/2}$.

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As we have seen, one description is as a Cayley graph on a dihedral group. It is easy to see that the connection set must consist of three reflections.

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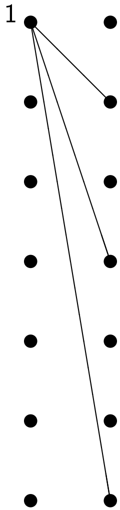

As soon as we know the neighbours of 1, the remaining edges arise by cyclically rotating the edges as displayed.

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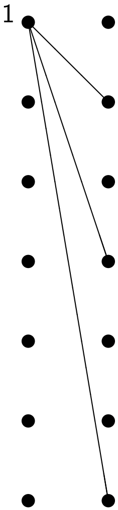

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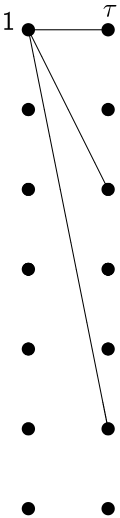

We may cyclically relabel one coset so that τ belongs to the connection set.

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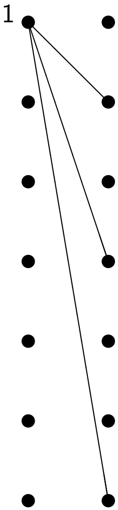

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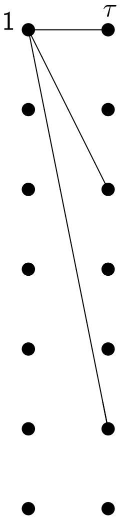

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If there is an invariant Hamilton cycle, it must be generated by two of the reflections. This means the difference ℓ between their jumps satisfies $\gcd(n/2, \ell) = 1$.

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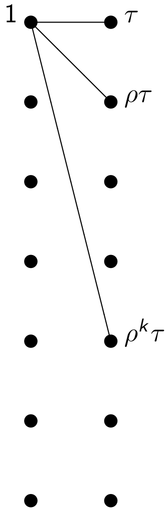

There is then an isomorph such that both τ and $\rho\tau$ are in the connection and generate the Hamilton cycle.

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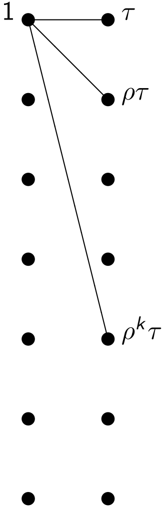

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Using some work of Coxeter, Frucht and Powers, we prove that the Cayley graph is $\mathcal{F}(1,2)$-invariant iff none of the following congruences hold:

$k^2 \equiv 1 (\text{mod } n/2)$

$(k-1)^2 \equiv 1 (\text{mod } n/2)$

$k(k-1) \equiv -1 (\text{mod } n/2)$

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We now move to the case that the 2-factor is composed of two n-cycles. So let X be such a graph with the two cycles being C_1 and C_2.

Let $H \leq \mathrm{Aut}(X)$ be the subgroup fixing $V(C_1)$ and $V(C_2)$ setwise. Because there is a perfect matching joining $V(C_1)$ to $V(C_2)$, it is clear that the restriction H_1 of H to $V(C_1)$ is faithful.

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Because H_1 is a transitive subgroup of D_n, it is possible it contains an n-cycle. If it does, then we may suppose that

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The preceding implies that X is a generalized Petersen graph. Recall that the generalized Petersen graph $\mathrm{GP}(n,k)$ has vertex set $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and the following edges: $[u_i, u_{i+1}], [u_i, v_i], [v_i, v_{i+k}]$, where $i = 1, 2, \ldots, n$ and subscript arithmetic is carried out modulo n.

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The following figure illustrates $\mathrm{GP}(6, 2)$. Also note that $\mathrm{GP}(5, 2)$ is the Petersen graph.

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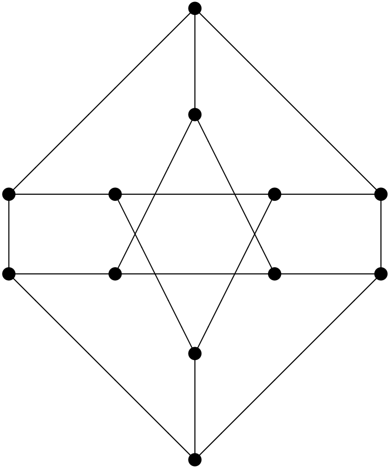

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The vertex-transitive generalized Petersen graphs and their automorphism groups were classified in the 1970s by Frucht, Graver and Watkins. From this we have that $\mathrm{GP}(n, k)$ is $\mathcal{F}(1, 2)$-invariant if and only if $k^2 \equiv \pm 1 (\mathrm{mod}\ n)$ except for (n, k) or $(n, n-k) \in \{(4, 1), (5, 2), (8, 3), (10, 3), (12, 5), (24, 5)\}$.

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This leaves us with the situation that H_1 does not contain an n-cycle. We then must have that $\rho^2 \in \mathrm{Aut}(X)$ together with an involution that makes H_1 transitive.

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How can we represent these graphs? Following is one way. The example is for $n = 10$.

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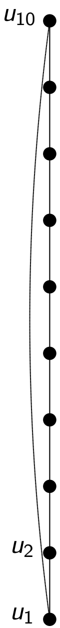

We may display one of the n-cycles as shown where ρ^2 is an automorphism of X.

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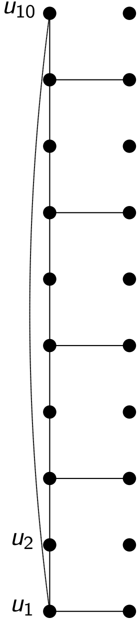

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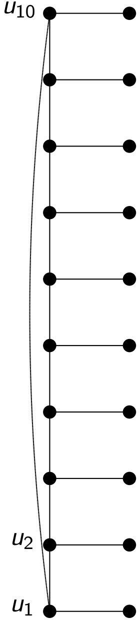

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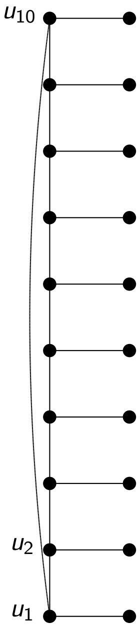

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The remaining edges of X arise from two odd integers $0 < k_1 < k_2 < n$, where v_i is adjacent to v_{i+k_1} and $v + i + k_2$ for all even i.

Thank You